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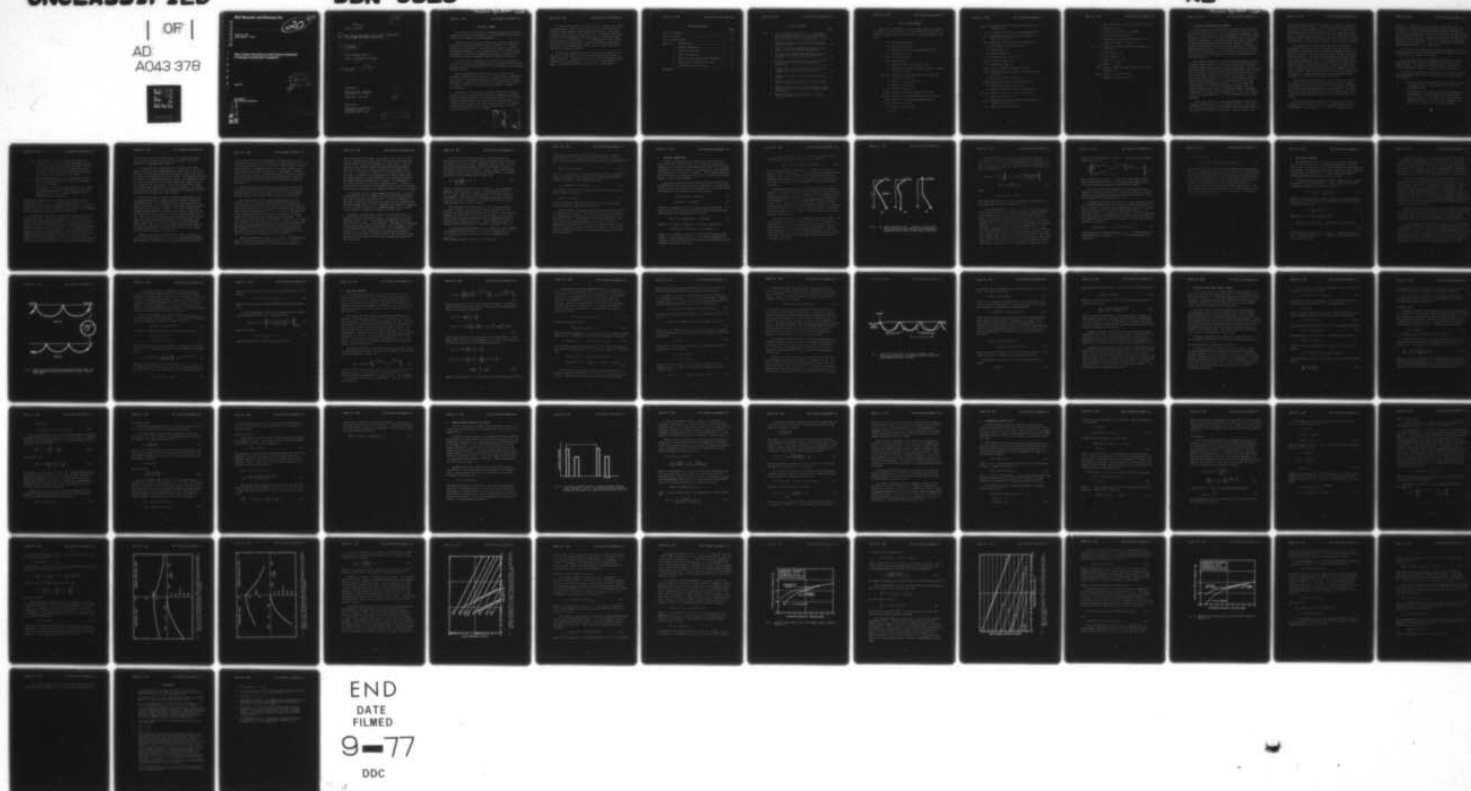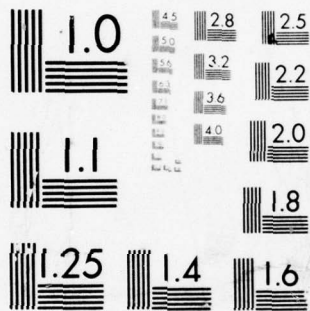JUN 77   P W SMITH, J C HEINE   N00173-77-C-0059

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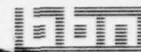
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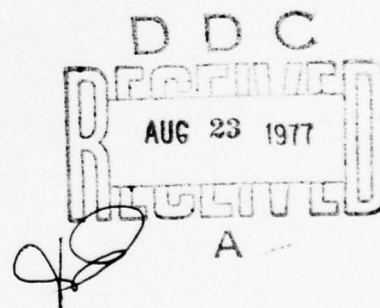
Report No. 3326
Interim Report — Task 3

Wave Theory Corrections to Ray Theory Predictions of Average Long-Range Propagation

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WAVE THEORY CORRECTIONS TO RAY THEORY PREDICTIONS OF AVERAGED LONG-RANGE PROPAGATION.

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J.C./Heine

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EXECUTIVE SUMMARY

This study of corrections to propagation predictions based on ordinary ray theory is motivated by the problem of low-frequency ambient noise caused by surface shipping.

Shipping noise is caused by many sound sources, widely distributed in range and azimuth from any receiver. The sources — propellers and machinery — are near the sea surface, at depths where large gradients of the sound speed are often found. These circumstances can lead to significant errors in calculations of low-frequency sound propagation based on ordinary ray theory.

The objective of this study is to quantify the significant errors in ray theory calculations, and to indicate when they arise and how corrections may be made in the calculation of low-frequency shipping noise.

The most significant error concerns the acoustic source strength. When the source ship is in a region having large negative values of sound-speed gradient, ordinary ray theory *underestimates* the sound power injected into ray bundles with shallow D/E angles. Errors of 6 dB and more are found in realistic circumstances.

Correction for this error is readily achieved by using an *effective source strength* that depends on frequency, D/E angle, and the sound speed profile in the surface layers. The effective source strength amounts to a wave theoretical modification to Lloyd's mirror formula, which has often been used in ray theory to describe the reduction in source strength due to a source being located near the ocean surface.

A second error concerns the path of rays. Ordinary ray theory *overestimates* the range interval for which shallow rays dwell in a surface layer having a large sound-speed gradient. The error occurs on every cycle to the surface, and is cumulative with range. In realistic circumstances, errors exceeding 0.5 NM every 35 NM can be made at low frequencies. As a consequence of this effect, ray theory calculations of the intensity transmitted from point to point can show significant errors, even if corrections have been made for source strength.

An important finding of this study is that the errors in ray path have no significant effect on the range-average of transmitted intensity. Therefore, they do not affect the ambient noise calculated for a distribution of ships over a broad interval of range. No correction need for made for this second error.

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LIST OF MAJOR SYMBOLS

Symbols are identified briefly in words and by the number, in brackets, of the equation in or near which they are first introduced. Roman letters preceed Greek; a capital preceeds lower case.

$$A = \text{ray tube area [33]}$$

$$Ai = \text{Airy function [64]}$$

$$B = \text{parameter related to speed gradient [58]}$$

$$Bi = \text{Airy function [64]}$$

$$C, C_1, C_2 = \text{constants [63,20]}$$

$$c(z) = \text{sound speed [3]}$$

$$c_v = \text{speed at depth of ray vertex [29]}$$

$$D_0 = \text{directional factor in effective source strength of monopole [44]}$$

$$D_{HF}, D_{VF} = \text{same, for horizontal force source [49] and vertical force [51]}$$

$$F = \text{force of dipole source [1]}$$

$$f = \text{frequency in Hz [59]}$$

$$G = \text{Green's function [4]}$$

$$G_n^+, G_n^- = \text{terms in multipath expansion of G [11]}$$

$$g, g_u, g_\ell = \text{solutions to separated wave equation [20,5]}$$

$$g^* = \text{negative of sound-speed gradient [59]}$$

$$H_0^{(1)} = \text{Hankel function [5]}$$

$$I = \text{sound intensity [33]}$$

$$i = (-1)^{\frac{1}{2}}$$

k, k_r, k_z = wavenumber and its radial and vertical components [3,5]

k_v = stationary value of k_r, a ray parameter [17]

$M(y)$ = magnitude of travelling wave solution to Airy's equation [63]

$N(y)$ = magnitude in Airy function analysis [81]

n = order number of multipath [10]

P = sound power [33]

p = sound pressure [1]

Q = volume velocity of source [1]

R, R_0 = position vectors [1]

r = horizontal range [4]

S, S_0 = source strength of monopole [39,42]

S_{eff} = *effective source strength of various sources* [44,49,51]

u = particle velocity vector [1]

V_u, V_ℓ = plane-wave reflection coefficients [5]

X = range period of ray path [32]

y = depth coordinate in Airy's equation [61]

y_0, y_1, y_2 = values of y at z = 0, z_1, z_2 [62, 68, 76]

z = depth coordinate [4]

z_1, z_2 = deeper and shallower depths, respectively, of observation points [4]

z_v = depth of turning point [22]

α = angle coordinate in horizontal plane [4,47]

γ = phase of g_u [14]

Δ indicates increment in a variable

δ = Dirac's delta function [1]

η = phase angle in Airy function analysis [81]

θ = depression angle of a ray [6]

θ_0 θ_1 θ_2 = values of θ at the surface and depths z_1,z_2
        [73,30,56]

ρ = fluid density [1]

Φ = total phase [16]

ϕ = phase of $(V_u V_\ell)$ [14]

ϕ_u = phase of V_u [66]

$\psi(y)$ = phase of travelling wave solution to Airy's
        equation [63]

ψ_0,ψ_1 = $\psi(y_0)$ [63] and $\psi(y_1)$ [68]

ω = angular frequency, $2\pi f$ [1].

1.   TECHNICAL INTRODUCTION AND SUMMARY

The accuracy of ray theoretical calculations is founded in large part upon a hypothesis that the fractional change of sound speed in a distance equal to the wavelength is sufficiently small. Obviously, the hypothesis becomes increasingly poor as the sound frequency diminishes. Furthermore, since the largest gradients of sound speed are observed near the sea surface in typical oceanic environments, the errors will be greatest for sound propagating near the surface. Here we explore the nature of those errors and the possibility of correcting for them, giving particular attention to the important case where the sound source is near the surface.

The physical problem of interest is that component of low-frequency noise observed in the deep ocean which is attributable to surface ships distributed over the oceanic basin. Surface ships can create sound by many different mechanisms, some of which may be idealized as simple sources (monopoles) and others as localized forces (dipoles) with various orientations. Some of the sound energy is refracted into the SOFAR channel and propagates to great distances; the rest interacts with the ocean bottom and is attenuated more rapidly. We shall ignore the latter component. The noise observable at any one point is caused by many ships, located at different ranges. Therefore, a prediction of point-to-point transmission is of less interest, in the end, than predictions valid for averages over the range variable.

The overall problem of long-range propagation from a low-frequency, near-surface source includes two parts. The first part concerns the strength of radiation that is observed deep in the ocean at a relatively short range. This strength varies

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with the depression angle (re horizontal) at which the sound is
observed to be propagating. In ray theory, this strength can be
evaluated by adding coherently two components: one direct from
the source and one from the image (i.e., the surface-reflected
path). A recent study by Pedersen *et al.* [1] indicates that this
strength may be poorly estimated by ray theory for some realistic
combinations of source depth, frequency, and sound speed gradient.

The second part of the problem concerns the subsequent
spreading of this energy as it propagates down the channel. In
ray theory, the energy travels along ray paths whose location is
governed by Snell's law, and the energy flux between neighboring
rays is conserved. A recent study by Murphy and Davis [2] indi-
cates that the ray paths and the distance between neighboring
paths are both distorted when low frequency sound passes through
regions having a large speed gradient. Hence, ray theory may
lead to significant errors in estimating the transmission from
point to point. However, we shall see in the present study that
this error is largely compensated for when the range-averaged
transmission is calculated by simple ray theory.

This study has two objectives. First, we want to quantify
the domain of situations where significant errors in shipping
noise predictions can arise from the use of ordinary ray theory
to calculate the sound generation and transmission. Second, we
wish to determine how corrections, based on wave theory, can be
introduced into ray theory calculations to compensate for those
errors.

The reader should particularly note two aspects of this
statement of objectives. First, it is assumed that the test of
significance of any errors is to be found in shipping noise

prediction. The peculiarities of ships (many shallow, low-frequency sources, widely distributed in range) strongly affect the balance of attention given different problems. Second, it is taken as a premise that ordinary ray theory calculation schemes will be retained, but that corrections can and will be made to the details of their procedures. Thus, we exclude the not entirely implausible view that, given the errors, a different calculation scheme should be used (e.g., modal series).

At this point, we summarize the analytical approach and the principal results for the technical reader who wishes to be spared the elaboration of details that follow.

It is assumed that the environmental parameters vary only with depth and that large speed gradients are limited to the surface layers. We consider transmission from a shallow point in the high-gradient layers to a deep point at long range.

The analytical formulation proceeds by these steps, familiar from Brekhovskikh [Ref. 4, Sec. 36] and many other analyses in the literature:

    i. The sound field is expanded into plane waves inclined at various angles to the horizontal at the deep observation point.

    ii. The separated wave equation governing the depth-wise distribution of sound pressure for a single plane wave is "solved", in the sense that the solution is explicitly defined. Procedures for evaluating the solution are put off for later consideration.

iii. The responses to the individual plane waves are
      superposed, yielding an integral expression for the
      total sound field at the shallow observation point.
      At a later stage of analysis, reciprocity will be
      invoked to justify identifying the shallow observa-
      tion point as the location of the source.

iv.   The multipath expansion of the integrand is intro-
      duced, so that the total sound field is expressed
      as a sum of integrals.

v.   The integrals of the sum are evaluated by the method
      of stationary phase when conditions are such that a
      stationary point exists; those integrals having no
      stationary point are dropped.

Brekhovskikh demonstrates that ordinary ray acoustic result
from using in step (ii) the ordinary WKB expression for the
solution to the separated Z-dependent wave equation. The sound
field is found as a sum of components identifiable as transmissions
along the many ray paths defined by Snell's law.

   The desired corrections to ordinary ray theory result from
using in step (ii) evaluation procedures that are more nearly
precise than ordinary WKB procedures. There are various alterna-
tives. One might use generalized WKB procedures in the manner
of Murphy and Davis in their development of a "modified ray theory"
[Ref. 2]. At one level of complexity, this procedure amounts to
representing the sound speed profile as one that is locally iso-
gradient with slowly changing gradient. The solution for the
sound field is correspondingly expressed in terms of Airy functions
(precise in a constant gradient) with slowly changing parameters.

Another alternative is to use numerical integration from the
ocean surface in order to evaluate the solution to the one-
dimensional, separated wave equation.

In any case, the requirement for a solution more nearly
precise that the ordinary WKB expression can be met by the appli-
cation of improved procedures to a restricted zone of the ocean,
including the high-gradient surface layers. The gradients observed
at depth in the ocean are smaller; moreover, the depression angle
of a ray originating at the shallow observation point increases
as it reaches into the depths. Both these facts tend to make
the ordinary WKB expression more accurate at depth. The errors
of ordinary ray acoustics in the shipping noise problem originate
when the rays are near the ocean surface.

Section 4 below constitutes a detailed examination of the
field associated with a single multipath integral at the deep
observation point. It is shown that the local sound field at
depth follows the laws of ordinary ray acoustics if the ordinary
WKB expression is *locally* accurate. (It is not assumed that
ordinary WKB procedures are valid at all depths.) Specifically,
the field lines of the normals to phase fronts satisfy Snell's
law. The locus of observation points having the same stationary
point (in the stationary-phase analysis of the integral over
radial wavenumber) is a normal to phase fronts and satisfies
Snell's law. The power in a ray bundle (defined as an interval
of radial wavenumber) is invariant along such a locus. These
are all laws of ordinary ray acoustics.

However, two errors are identified. First, the path of a
ray (the locus defined above) is not that predicted by ordinary
ray acoustics. Specifically, an error is made every time that

a ray cycles through the high-gradient surface layers, a result
carefully delineated in the prior work of Murphy and Davis [Ref. 2].
The range-wise period ("cycle length") of such ray paths is incor-
rectly evaluated. Therefore, the error in path location grows
with range in multicycle transmission to long ranges. Moreover,
since this error in range period varies from ray to ray, ordinary
ray theory inaccurately assesses the ray-tube width, leading to
errors in sound intensity although not affecting the power in a
ray bundle.

The second error of ordinary ray theory arises in the cal-
culation of sound power in a ray bundle, when and only when an
observation point (e.g., the source) lies in the high-gradient
surface layers. This leads to more errors in sound intensity.

The second error is very simply compensated for by the
introduction of an *effective source strength*, defined as the
source strength which yields the correct value of sound power in
a ray bundle when used in the formulas of ordinary ray acoustics.
These ideas are developed in Sec. 5. The value of the effective
source strength is found from the precise solution at shallow
depths of the separated, one-dimensional wave equation. Just
like Lloyd's mirror formula, it includes the effect of the coherent
sum of what in ordinary ray acoustics are up-going and down-going
rays with the same depression/elevation angles at the source.
The strength of the sum is associated with one down-going ray.
The result differs both from Lloyd's mirror formula and from the
result of strict ray tracing with coherent addition.

Next, an examination is made of the effect of averaging over
range the transmitted squared-pressure (Sec. 6). It will be
recalled that the shipping noise problem involves many ships,

widely distributed in range. Even though calculation of point-
to-point transmission may be used as a numerical procedure, the
desired answer involves an average over many ranges. We have
adapted to the present analysis a range-averaging procedure
previously used in the context of ordinary ray theory [Smith,
Ref. 13]. A statistical estimate of the contribution of each
ray bundle is formed on the hypothesis that every range within
one range period is equally probable. The resulting expression
for range-averaged transmission is found to be negligibly af-
fected by the errors in ray path, described above. However, it
is essential that corrections be made for the errors in the power
injected into ray bundles, by using the "effective source strength."

We observe that the range-averaged transmission, corrected
by the effective source strength, is easily calculated directly
from the analytical expression for it. It is not necessary to
make many, erroneous calculations of point-to-point transmission
and then average those results. Moreover, this analytic averaging
procedure could be adopted to a range-dependent ocean by means
of the adiabatic invariance of action [cf. Ref. 13].

Numerical and analytical results for the errors and cor-
rection factors are developed in Sec. 7 for sources in an iso-
gradient surface layer. The results are based on the precise
wave solution in terms of Airy functions. Throughout the analysis,
a depression angle computed from Snell's law is frequently used
as the parameter that identifies a ray. It is especially con-
venient to do so when talking about corrections to ordinary
ray theory, even though the angle does not have the same physical
meaning as in ray theory when the wave-theoretical corrections
are significant.

The error in the range period of a ray varies with the depression angle at the surface, being a maximum for the ray that is horizontal at the surface. Significant errors exist for rays that vertex below the surface (within a wavelength) as well as for those that strike the surface. Figures 5 and 6 show results for typical values of low frequency and high gradient. The maximum error, for the grazing ray, is given by*

$$\Delta X = \frac{1.26}{f^{1/3}_{Hz} \; g^{*2/3}_{sec^{-1}}} \; km \quad , \tag{74}$$

where f is frequency and g* is the negative of the sound-speed gradient, hence a positive number. As discussed in detail in Sec. 7, this formula is not applicable for very small values of the gradient. However, within its range of validity, errors in excess of 1 km are predicted for frequencies of 200 Hz or less and gradients of -0.1 sec^{-1} or more.

The effective source strength of a unit monopole source is found to exceed that given by ordinary ray theory by 6 dB and more for small depression angles at the source. For realistic numbers (f = 25 Hz, g* = 0.5 sec^{-1}, source depth = 20 ft), the error starts to grow significantly when the depression angle is reduced below 8 degrees (Fig. 8). Larger errors are found for smaller values of the ratio of frequency to gradient (Fig. 9).

An analytical approximation for the effective source strength of a monopole has been found. For steeply inclined rays, the well-known Lloyd's mirror formula is indistinguishable from

---

*The equation numbers are those of the main text.

ordinary ray theory and from more accurate results. Lloyd's
mirror formula for a shallow source whose strength is unity in
free space yields a directional source strength given approximately
by

$$D_0^{LM} \approx 158(z/\lambda)^2 \sin^2\theta \, , \qquad\qquad (78)$$

where z is source depth, λ is the wavelength, and θ is depression
angle at the source. By contrast, the Airy-function analysis
yields an effective source strength given by

$$D_0 \approx 68(g*/f)^{1/3} (z/\lambda)^2 \sin\theta \, . \qquad\qquad (77)$$

This formula is valid in the domain where it exceeds the Lloyd's
mirror result, that is for

$$g*/f > 12.5 \sin^3\theta \, . \qquad\qquad (79)$$

This is the domain where both ordinary ray theory and Lloyd's
mirror formula are significantly in error.

Similar errors of ordinary ray theory are found in the case
of dipole (force) sources. The details, which vary according to
the orientation of the force (vertical, horizontal, etc.), are
developed in Sec. 7. The domain in which wave-theoretical cor-
rections to the effective source strength must be used is
approximately the same as Eq. 79 for the monopole source. More-
over, the correction factor that must be applied to the Lloyd's
mirror formula for a dipole is approximately the same as in the
monopole problem.

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2.   INTEGRAL FORMULATION

We are concerned with sound propagation from pure-tone volume and force sources (monopoles and dipoles) located in an inhomogeneous fluid whose sound speed $c(z)$ varies with the depth coordinate z and whose density ρ is constant [3]. All properties are independent of horizontal position. Using complex variables, we include the time dependence implicitly as a factor $\exp(-i\omega t)$, where $\omega = 2\pi f$ is the frequency.

The sound pressure $p(R)$ and particle velocity vector $u(R)$, observable at a vector position R in response to a volume source Q and a vector force source F both located at $R = R_0$, must satisfy the equations of continuity and of force balance:

$$-i\omega c^{-2} p + \rho \nabla \cdot u = \rho Q \delta(R-R_0) \tag{1}$$

$$-i\omega \rho u + \nabla p = F \delta(R-R_0) \quad , \tag{2}$$

where $\delta(\cdots)$ indicates a three dimensional delta function. The particle velocity is readily eliminated, with an inhomogeneous wave equation for the pressure as the result:

$$\nabla^2 p + k^2 p = i\omega\rho Q \delta(R-R_0) - F \cdot \nabla_0 \delta(R-R_0) \quad , \tag{3}$$

where $k(z) = \omega/c(z)$. We have used the vector identities

$$\nabla \cdot [F\delta(R-R_0)] = F \cdot \nabla\delta(R-R_0) = -F \cdot \nabla_0 \delta(R-R_0) \quad ,$$

where F is a constant vector and ∇_0 denotes the gradient with respect to the source location R_0. The complete mathematical statement of the problem requires additional specification of the boundary conditions; see below.

The field in response to either source is readily derived from the Green's function, defined as the solution to

$$[\nabla^2 + k^2(z)] \, G(R,R_0) = -4\pi\delta(R-R_0) \quad , \tag{4}$$

with the same boundary conditions. This Green's function approaches $|R-R_0|^{-1}$ as $|R-R_0| \to 0$, and thus has the standard normalization to the field at a unit range. Under broad conditions, which we assume to hold here, G is reciprocal, in the sense that its value is invariant to an interchange of source and receiver locations.

Cylindrical coordinates (r,z,α) are convenient for further analysis. We take the axis r=0 to be a vertical line through the source. The origin of depth is placed at the sea surface, with z increasing downward. The source depth is z_1. The solution for G is independent of α. A typical sound speed profile is sketched in Fig. 1(a). In this analysis, the source is deep; we shall seek solutions for points near the surface, relying on reciprocity to justify using the solution for the actual problem of interest with the source near the surface. The sea surface is taken as a boundary at which G=0.

The boundary condition at the bottom will be left vague. We may anticipate that it plays only a minor role in long-range propagation since we are interested only in ducted energy: i.e., rays that do not touch the bottom, or trapped modes. Bottom properties must be specified in order numerically to perform a modal analysis, but to the small extent that they affect results the effect is probably erroneous due to artificiality of the modeling of the bottom.
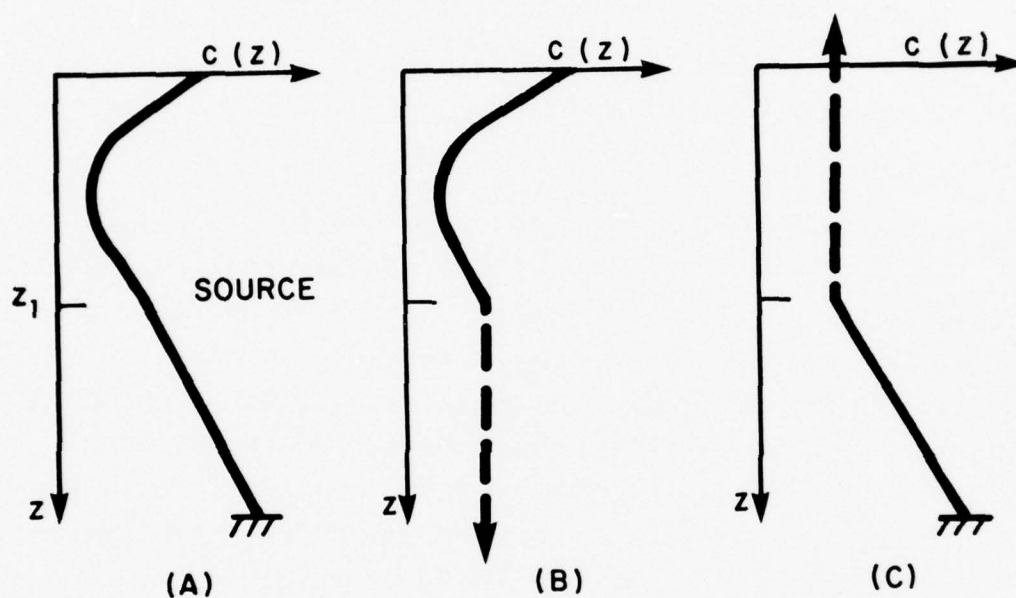
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FIG. 1. (A): ACTUAL SOUND SPEED PROFILE. (B) AND (C): SEMI-INFINITE
PROFILES USED IN AUXILIARY PROBLEMS TO DEFINE REFLECTION
COEFFICIENTS FOR THE UPPER AND LOWER SPACES, RESPECTIVELY.

Brekhovskikh has shown that the solution for G can be expressed as a superposition of cylindrical waves diverging from the source [5]. Each wave is characterized by its value of the radial wavenumber k_r, and the formal solution is an integral over all real values of k_r:

$$G(z_2,r,z_1) = \frac{i}{2} \int_{-\infty}^{\infty} \left[g_{u,\ell}(z_2,z_1,k_r) \frac{1 + V_{\ell,u}(z_1,k_r)}{1 - V_\ell V_u} \right] \times$$

$$H_0^{(1)}(k_r r) \frac{k_r}{k_z(z_1)} dk_r , \tag{5}$$

where

$$k_z(z) = [k^2(z) - k_r^2]^{\frac{1}{2}} . \tag{6}$$

The first subscript of the pairs appearing in the integrand is used when $z_2 \leq z_1$ and the second for $z_2 \geq z_1$.

The terms appearing in brackets in Eq. 5 (g_u, g_ℓ, V_u, V_ℓ) are found from solution of two auxiliary problems: the reflection of an incident plane wave of unit amplitude from either the upper (u) or lower (ℓ) half of the channel ($z \leq z_1$ or $z \geq z_1$). Consider the reflection analysis of the upper half. The actual channel for $z < z_1$ is supplemented by a semi-infinite homogeneous halfspace, $z_1 \leq z < \infty$, with constant sound speed $c(z_1)$. The resulting sound speed profile is shown in Fig. 1(b). A plane wave is generated in the homogeneous halfspace, traveling upward at an angle θ with respect to horizontal, where $\tan\theta = k_z(z_1)/k_r$. This incident wave has unit amplitude and is phased to have zero phase angle at z_1. Reflection of the incident wave will cause response in the channel and a reflected wave traveling at an

angle $-\theta$ in the homogeneous halfspace. Thus, the total pressure field has the form

$$
p = \left\{ \begin{array}{l} \left[e^{-ik_{z_1}(z-z_1)} + V_u \, e^{ik_{z_1}(z-z_1)} \right] e^{ik_r x} \, , \quad z \geq z_1 \\[2ex] g_u(z) \, e^{ik_r x} \, , \quad z \leq z_1 \end{array} \right\} \tag{7}
$$

where we write $k_{z_1} = k_z(z_1)$. Hence, V_u is the complex plane-wave reflection coefficient for the upper half channel and g_u is the corresponding response inside the half channel.

The terms g_ℓ and V_ℓ are defined by a similar reflection problem in which the lower half of the actual channel is supplemented by a semi-infinite homogeneous halfspace above it. This sound speed profile is shown in Fig. 1(c).

A mathematician might take another, totally equivalent view. In a straightforward derivation of Eq. 5, the Hankel transform of Eq. 4 is taken to eliminate the r-dependence, yielding an ordinary differential equation in z with a source term $-2\delta(z-z_1)$. Its solution, which is the bracketed term in Eq. 5, can be constructed from the solutions to boundary-value (initial-value) problems for the upper and lower halves of the channel. For the upper half we require the solution to

$$
g_u'' + k_z^2 \, g_u = 0 \, , \qquad k_z^2 = k^2(z) - k_r^2 \, , \tag{8}
$$

with appropriate boundary conditions at z=0. (The prime denotes differentiation with respect to z.) The initial value at $z=z_1$ is defined as

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$$g_u(z_1) = 1 + V_u$$

$$V_u = [(-ik_z - g_u'/g_u)/(-ik_z + g_u'/g_u)]_{z=z_1} \quad . \tag{9}$$

The functions g_ℓ, V_ℓ are found from a similar initial-value problem for the lower half of the channel, except that in the definition of the reflection coefficient, Eq. 9, ik_z is substituted for $-ik_z$. The desired solution to the inhomogeneous equation (with source term) is represented by g_u and g_ℓ when they are scaled to satisfy the appropriate continuity conditions across $z=z_1$ (continuous function and discontinuous first derivative). Then the inverse Hankel transform leads to Eq. 5, when symmetry of the integrand is invoked to extend the lower limit of the integral to $-\infty$ (cf. Ref. 4, p. 243).

3.   MULTIPATH EXPANSION

Brekhovskikh has shown how the general integral expression for the sound field, Eq. 5, is readily expanded into a sum of terms, each of which can be shown, with appropriate approximations, to represent the contribution of energy travelling along a distinct ray path from the source [6].   In the present development we follow Brekhovskikh, with minor generalizations.

The essential step toward a multipath expansion is to expand the factor in the brackets of Eq. 5 into a power series in the reflection coefficients, using the identity

$$[1 - V_u V_\ell]^{-1} = \sum_{n=0}^{\infty} (V_u V_\ell)^n \quad , \tag{10}$$

which is convergent in any physical case since slight dissipation causes $|V|$ to be absolutely less than unity.   The Green's function then has the form

$$G = \sum_{n=0}^{\infty} [G_n^+ + G_n^-] \quad , \tag{11}$$

where (for $z_2 < z_1$) the typical term is

$$G_n^+(z_2, r, z_1) = (i/2) \int_{-\infty}^{\infty} g_u \, V_u^n V_\ell^n \, H_0^{(1)}(k_r r) \, [k_r/k_z(z_1)] dk_r \quad . \tag{12}$$

The integrand for G_n^- differs in having V_ℓ raised to the power n+1.   The expressions for $z_2 > z_1$ differ in having the subscripts u and ℓ interchanged.

Brekhovskikh shows, for a special case, that the term G_n^+ corresponds to transmission along the ray path that leaves the source in an upward direction and loops n times into both the upper and lower halves of the channel, being turned each time by reflection or refraction. The path corresponding to G_n^- leaves the source in a downward direction, being turned n+1 times in the lower half and n times in the upper half channel. (See Fig. 2.)

In the case Brekhovskikh chooses for illustration of the technique, the boundary z=0 is rigid and the receiver lies on the boundary. However, the generalization is apparent. Within the WKB approximation, the field at the receiver point can be split into upward and downward travelling parts, whose inter- ference is described by the function g_u. If G_n^+ were correspond- ingly split into two integrals, each would be found to describe a different ray path. One of the paths would approach the receiver from below and the other from above, after reflection from the surface (see insert in Fig. 2).

However, in the cases of present interest, where the receiver is near the surface and at a long range, it would be inappropriate to attempt such a decomposition of G_n. The two parts are not resolved in any measurements of interest. The analysis must retain the coherent interaction between the parts, as in Eq. 12.

The correspondence between the integral for G_n and the pres- sure transmitted along the ray path is established by (1) using expressions for g_u, V_u, V_ℓ based on the WKB approximation to solu- tions of Eq. 8, and (2) evaluating the integral by the method of steepest descents in the complex k_r plane. The resulting values of amplitude and phase are found to be the same as in ordinary ray theory, except for a negligible correction to the phase.

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SOURCE   RECEIVER
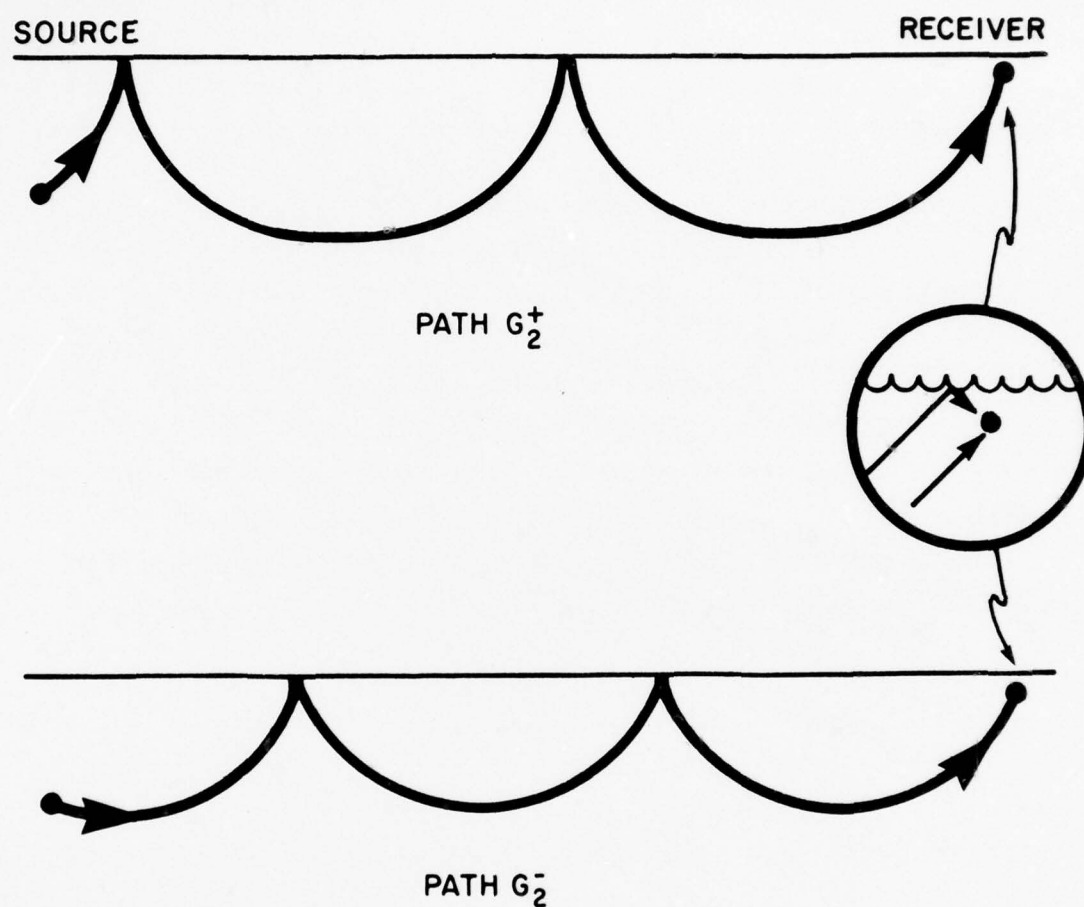
PATH G_2^+

PATH G_2^-

FIG. 2. TYPICAL RAY PATHS CORRESPONDING TO INDIVIDUAL MULTIPATH TERMS. THE
        INSERT INDICATES THE SURFACE INTERFERENCE EFFECT WHICH IS INCLUDED
        IN EACH TERM.

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In the present study we hypothesize the failure of the of the ordinary WKB approximation in the near-surface layers (where the receiver point is located), as a consequence of a low frequency and large speed gradient. However, we hypothesize the validity of the WKB approximation near the source ($z = z_1$) and, indeed, throughout the depth of the ocean except the surface layers. We follow Brekhovskikh's procedure, insofar as possible, and develop corrections to ordinary ray theory.

First let us develop the formal expression for G_n, Eq. 12, using the method of stationary phase. We can justify a posteriori using the asymptotic form for the Hankel function:

$$H_0^{(1)}(z) \sim (2/\pi z)^{\frac{1}{2}} e^{i(z - \pi/4)} \quad . \tag{13}$$

The phases of other terms in the integrand are denoted by:

$$\arg V_u V_\ell = \phi(z_1, k_r)$$

$$\arg g_u = \gamma(z_2, z_1, k_r) \quad . \tag{14}$$

[In the case of a term G_n^-, γ would denote $\arg(g_u V_\ell)$.] We take as the value of the integral that given by the method of stationary phase:

$$\int A(x) e^{i\Phi(x)} dx \approx \left[\frac{2\pi}{|\Phi''(x_0)|} \right]^{\frac{1}{2}} A(x_0) e^{i[\Phi(x_0) \pm \pi/2]} \quad , \tag{15}$$

where x_0 is the point of stationary phase where the derivative Φ' vanishes, and the choice of sign in the exponent agrees with the sign of Φ'' [7]. The total phase of the integrand in Eq. 12 is

$$\Phi = k_r r + n\phi + \gamma - \pi/4 \quad , \tag{16}$$

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which has a stationary point at the root, denoted by k_v, of the equation

$$0 = r + n\phi' + \gamma' \quad \text{at } k_r = k_v \quad , \tag{17}$$

where the prime indicates partial differentiation with respect to k_r.

The straightforward combination of these relations yields for the squared magnitude of the multipath term:

$$|G_n^+(z_2, r, z_1)|^2 = \left[\frac{|g_u(z_2, z_1)|^2 \ |V_u V_\ell|^{2n} \ k_r}{r \ k_z^2(z_1) \ |\Phi''|} \right]_{k_r = k_v} \tag{18}$$

and for its phase:

$$\arg G_n^+ = \Phi_{k_r = k_v} + \pi \pm \pi/2 \quad , \tag{19}$$

with the sign chosen to agree with the sign of Φ''.

4.   LOCAL RAY BEHAVIOR

In this section we examine the spatial dependence of the preceeding expressions for the contribution of one multipath term in order to find the similarities and differences with the results of ordinary ray acoustics. The physical discussion is simpler if we once more appeal to reciprocity: we shall call the near-surface point at z_2 the "source", and the deep point at z_1 the "receiver".

The results of this section are intimately related to the validity of the WKB approximation to the solutions g_u and g_ℓ of the z-dependent, one-dimensional wave equation (Eq. 8). The local variations of sound field may be well described by the WKB solution in some intervals of z, which we call "WKB regions". The WKB solution may be inaccurate in other intervals, called "non-WKB regions". In what follows, we assume that the surface layers, containing the source at z_2, constitute a non-WKB region, while the rest of the ocean, containing the receiver at z_1, is a WKB region. The results could readily be generalized to cases with several non-WKB regions [8].

The WKB approximation represents the general solution to the one-dimensional wave equation as the sum of components travelling upward and downward, in the form

$$g(z) = [k_z(z)]^{-\frac{1}{2}} \left(C_1 e^{-i\int^z k_z dz} + C_2 e^{+i\int^z k_z dz} \right) , \quad (20)$$

where C_1 and C_2 are arbitrary constants [9]. In the case of the function g_u (Eq. 8), the constant C_1 is by definition to be chosen so that the upgoing wave is unity at $z=z_1$. The constant C_2 is related to the reflection coefficient. Then the solution can be written

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$$g_u(z,z_1) = \left\{ \left[\frac{k_z(z_1)}{k_z(z)} \right]^{\frac{1}{2}} e^{-i\int_{z_1}^{z} k_z dz} \right\} \left[1 + V_u(z_1) e^{i2\int_{z_1}^{z} k_z dz} \right] ,$$

(21)

for any two points z and z_1 both in the WKB region. If the WKB region were to extend all the way to the surface $z=0$ and $k_z^2(0)>0$, then it can easily be shown that

$$V_u(z_1) = -\exp\left[i2\int_0^{z_1} k_z dz \right]$$

$$g_u(z,z_1) = -i2\sin\left[\int_0^{z} k_z dz \right] \left| k_z(z_1)/k_z(z) \right|^{\frac{1}{2}} \exp\left[i\int_0^{z_1} k_z dz \right] .$$

(22a)

This corresponds to a ray that *reflects from the surface*. On the other hand, if $k_z^2(0)<0$ corresponding to a ray that vertexes below the surface, then the ordinary WKB expressions are

$$V_u(z_1) = -\exp i2\left[\frac{\pi}{4} + \int_{z_v}^{z_1} k_z dz \right]$$

$$g_u(z,z_1) = i2 \sin\left[\frac{\pi}{4} + \int_{z_v}^{z} k_z dz \right] \left[k_z(z_1)/k_z(z) \right]^{\frac{1}{2}} \times$$

$$\exp i\left[\frac{\pi}{4} + \int_{z_v}^{z_1} k_z dz \right] ,$$

(22b)

where z_v is the depth of the turning point, defined by $k_z(z_v)=0$ [10].

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Now we wish to find the dependence upon z_1 of $g_u(z_2, z_1)$ when z_1 is in the WKB region but z_2 is not. The function g_u is by definition the response of a fixed system to a unit upgoing wave at z_1; its dependence upon z_1 arises solely from the necessity to renormalize to a unit upgoing wave when z_1 is changed. Therefore the dependence of g_u upon z_1 is completely given by the factor in braces in Eq. 21, which describes the upgoing wave in the WKB region. In particular, the phase $\gamma = \arg g_u$ satisfies

$$\partial \gamma(z, z_1, k_r) / \partial z_1 = k_z(z_1) \quad , \qquad (23)$$

and the magnitude satisfies

$$\partial [|g_u|^2 / k_z(z_1)] / \partial z_1 = 0 \quad . \qquad (24)$$

But, by the same argument, the last factor of Eq. 21 must be independent of z_1. Therefore, the reflection coefficient satisfies

$$\partial |V_u| / \partial z_1 = 0 \quad , \quad \partial \arg V_u / \partial z_1 = 2 k_z(z_1) \quad . \qquad (25)$$

When the same argument is carried out for the lower half channel, the signs in both exponents of Eq. 21 must be changed. It follows that

$$\partial |V_\ell| / \partial z_1 = 0 \quad , \quad \partial \arg V_\ell / \partial z_1 = -2 k_z(z_1) \quad ,$$

$$\partial |V_u V_\ell| / \partial z_1 = 0 \quad , \quad \partial \phi / \partial z_1 = 0 \quad \text{where } \phi = \arg V_u V_\ell \quad . \tag{26}$$

We are now in a position to examine the variations with receiver location (coordinates r and z_1) of the expressions given in Eqs. 18 and 19 for the strength of a single multipath term.

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Many similarities and some differences are found in comparison with the corresponding results of ordinary ray acoustics.

Consider the locus of positions having the same value k_V of the radial wavenumber at the stationary phase point. Equation 17 determines k_V as a function of r and other variables; its inversion yields an expression for r as a function of k_V:

$$r(z_1, z_2, k_V, n) = -\partial[n\phi(k_V) + \gamma(z_1, z_2, k_V)]/\partial k_V \quad . \tag{27}$$

The partial derivative with respect to z_1 is readily found, using Eq. 23:

$$\partial r/\partial z_1 = -\partial^2 \gamma/\partial z_1 \partial k_V = -\partial k_z(z_1, k_V)/\partial k_V = k_V/k_z(z_1) \quad ,$$

$$\tag{28}$$

since $k_z^2 = k^2 - k_V^2$ by definition. But this is just the equation for an ordinary ray path governed by Snell's law with a vertexing sound speed

$$c_V = \omega/k_V \quad . \tag{29}$$

The tangent to the locus is depressed from the horizontal by an angle $\theta_1(z_1)$ that satisfies

$$k_V = k(z_1) \cos\theta_1 \quad ,$$

$$k_z = k(z_1) \sin\theta_1 \quad . \tag{30}$$

Moreover the field lines of the gradient of phase (Eq. 19) describe the same locus, for it is readily shown from Eqs. 16, 23, and 26 that

$$\partial\phi/\partial r = k_V \quad , \qquad \partial\phi/\partial z_1 = \partial\gamma/\partial z_1 = k_z(z_1) \quad . \tag{31}$$

However, this ray-like behavior is valid only for local variations of z_1 within the WKB region. Equations 23 and 26 for the z_1-dependence of the phases γ and ϕ both fail in a non-WKB region. Equation 27 indicates that the locus of constant k_v is periodic in range with a period

$$\Delta r/\Delta n = -\partial\phi(k_v)/\partial k_v \triangleq X(k_v) \quad , \tag{32}$$

but this range period ("cycle length" or "bounce distance") is not correctly predicted by WKB analysis. Moreover, the range found by ordinary ray (WKB) analysis has an additional error due to the non-WKB behavior of γ. The locus of constant k_v and the ordinary ray path with c_v chosen as in Eq. 29 are sketched in Fig. 3. The error increases with increasing range because of the error in the range period X. The sketch exaggerates the effect. Calculations for typical cases made by Murphy and Davis [11] show errors of the order of 1 km or less in a range period X which, in the deep ocean, is typically about 70 km.

In summary, we have found that, within the WKB region, the locus of constant k_v is the normal to the phase fronts, and is the same as the path of the ordinary ray having vertex speed $c_v = \omega/k_v$ except for a range-wise displacement. The error increases with range because the range period X differs between the two paths.

Consider next the spatial variations within the WKB region of the squared magnitude of pressure, as given by Eq. 18. We first calculate the power associated with the sound field in a ray bundle, where "ray" is interpreted as a locus of constant k_v. Because these loci are equivalent to ordinary ray paths in the WKB region, the calculation procedure parallels that of ordinary
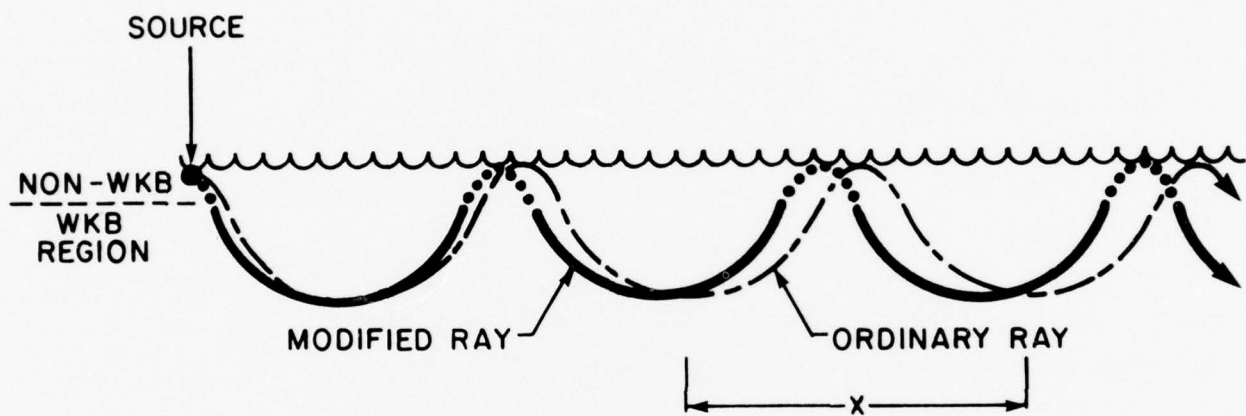
FIG. 3.   SKETCH OF PATHS OF MODIFIED RAY (LOCUS OF CONSTANT k_v) AND
          CORRESPONDING ORDINARY RAY.   THE MODIFIED RAY PATH IS NOT DEFINED
          IN THE NON-WKB REGION NEAR THE SURFACE.

ray theory, where the power in a ray bundle is found to be constant. The power P satisfies

$$\partial P/\partial k_V = I(k_V) \ \partial A/\partial k_V \quad , \tag{33}$$

where I is the acoustic intensity and A is the transverse area of a ray bundle δk_V measured in a direction normal to the rays. The intensity is related to the squared pressure:

$$I = |G_n|^2/\rho c(z_1) = k(z_1)|G_n|^2/\rho\omega \quad . \tag{34}$$

(This formula yields the peak intensity, not the time-averaged value which is half as large. Throughout this report we use peak intensity, peak power, and peak squared pressure. The formulas relating them are, of course, identical to the formulas relating the time-averaged quantities.) The transverse area is found by projection of the range derivative onto the normal to the rays:

$$\partial A/\partial k_V = 2\pi r \ |\partial r/\partial k_V| \sin\theta_1$$

$$= 2\pi r[k_z(z_1)/k(z_1)] \ |\partial r/\partial k_V| \quad , \tag{35}$$

where the factor $2\pi r$ arises from the cylindrical symmetry of the sound field about a vertical axis through the source.

Consistently with our interest in ducted ray paths, we assume

$$|V_u V_\ell| = 1 \quad . \tag{36}$$

It follows directly from Eqs. 16 and 27 that the range derivative is

$$\left| \partial r / \partial k_V \right| = \Phi''(k_V) \quad , \tag{37}$$

which is a factor in $\left| G_n \right|^2$. The straightforward combination of these results yields

$$\frac{\partial P}{\partial k_V} = \frac{2\pi}{\rho\omega} \frac{k_V \left| g_u(z_2, z_1, k_V) \right|^2}{k_z(z_1)} \quad , \tag{38}$$

where z_2 is source depth and z_1 is receiver depth. However, we have already found (Eq. 24) that this expression is independent of z_1 so long as z_1 is in the WKB region. In brief, the energy flux in a bundle of modified rays is invariant with position [12], just as in ordinary ray acoustics.

However, ordinary ray acoustics does not correctly predict this power. The factor $\left| g_u \right|^2$ is not correctly predicted when the source at depth z_2 lies in the non-WKB region. The error is independent of receiver depth z_1 (see Eq. 24) and independent of range (it is the same for each multipath term, independent of n).

Ordinary ray theory will lead to other serious errors in the prediction of squared pressure transmitted to a specific receiver point. First, the wrong ray (the wrong value of k_V) will be identified as connecting two points, because of the errors in ray path elucidated above. Second, since the errors in ray path vary as k_V changes from path to path, the wrong value of the range derivative will be calculated from ordinary ray acoustics. Both errors affect the magnitude of pressure.

5.   EFFECTIVE DIRECTIONAL SOURCE STRENGTH

In the last section, we showed that the sound power in a ray bundle is invariant with depth and range if the observation point is in the WKB region. (Dissipative processes are ignored in this analysis.) However, we found that ordinary ray theory does not correctly evaluate that power when the source is in a non-WKB region like the high-gradient surface layer.

In this section, we develop expressions for an *effective (directional) source strength*, defined as the source strength which yields the correct value of sound power in a ray bundle when the spreading is calculated by ordinary ray acoustics. Thus, the modification corrects for the errors in ordinary ray acoustics. The effective source strength includes the coherent sum of what in ordinary ray theory would be the direct and the surface reflected (or refracted) rays. This total strength is associated with a single, down-going ray at the source.

Different expressions are required for monopole sources (fluctuating volumes) and for dipole sources (fluctuating forces).

Ray Bundle Power in Ordinary Ray Acoustics

Consider a monopole source with source strength S, where "source strength" has the conventional meaning of the squared pressure observed at a unit range from the source in an infinite medium. We wish to find the expression for power in a ray bundle according to ordinary ray acoustics. Since this power is everywhere invariant with range, the calculation is most easily performed at a unit range near the source, where the sound is spreading spherically.

For the source of strength S located at depth z_2, the intensity at a unit range is just

$$I = S/\rho c(z_2) = S\, k(z_2)/\rho\omega \quad , \tag{39}$$

independent of the ray direction. Now consider a ray bundle where the rays are identified by the ray invariant

$$k_V = \omega/c_V \quad ,$$

where c_V is the vertexing sound speed. The relationship to the local depression angle θ of the ray is given by Snell's law

$$k_V = k(z)\,\cos\theta(z) \quad ; \tag{40a}$$

we shall also use the vertical component of wavenumber defined by

$$k_z(z) = k(z)\,\sin\theta(z) \quad . \tag{40b}$$

The differential of power dP in a ray bundle dk_V satisfies

$$dP/dk_V = I\, dA/dk_V \quad ,$$

where dA is the corresponding differential of area on the unit sphere:

$$dA = 2\pi\,\cos\theta\, d\theta = (2\pi\,\cos\theta/k\,\sin\theta)dk_V \quad .$$

The combination of these results with Eq. 39 yields the desired result:

$$\frac{dP}{dk_V} = \frac{2\pi}{\rho\omega}\,\frac{k_V S}{k_z(z_2)} \quad . \tag{41}$$

This expression gives the power in a ray bundle radiated from a source of strength S, according to ordinary ray acoustics.

We now compare the results of the earlier analysis which includes the effects of both surface interaction ("reflection") and wave-theoretical corrections.

Monopole Source

Consider a monopole source whose intrinsic source strength is S_0. By this it is meant that a squared pressure equal to S_0 would be observed at a unit range if the source were located in an infinite homogeneous medium. The Green's function G of Eq. 4 is normalized so as to have $S_0 = 1$. On the other hand, a monopole whose volume velocity is Q has an intrinsic source strength

$$S_0 = |\omega \rho Q / 4\pi|^2 \quad , \tag{42}$$

as is readily seen by comparison of Eqs. 1 and 3.

The power radiated into a ray bundle by a monopole of unit source strength was derived in the preceding section, Eqs. 33 through 38. For a monopole of strength S_0, the result is

$$\frac{\partial P}{\partial k_v} = \frac{2\pi S_0}{\rho \omega} \frac{k_v |g_u(z_2, z_1, k_v)|^2}{k_z(z_1)} \quad , \tag{43}$$

where z_2 is source depth and the expression has been shown to be independent of receiver depth, z_1. A comparison of Eq. 43 with the corresponding expression for ordinary ray acoustics, Eq. 41, shows that the *effective source strength* of the monopole is

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$$S_{eff} = S_0 D_0 \quad ,$$

$$D_0 = |g_u(z_2, z_1, k_v)|^2 \, k_z(z_2)/k_z(z_1) \quad . \tag{44}$$

When the environment is such that the WKB solution is accurate throughout the ocean, the directional factor D_0 reduces to familiar expressions of ordinary ray theory. From the WKB expressions for g_u given in Eq. 22, we find

$$D_0^{WKB} = 4 \, \sin^2 \left[\int_0^{z_2} k_z dz \right] \tag{45a}$$

when $k_z^2(0) > 0$, and

$$D_0^{WKB} = 4 \, \sin^2 \left[\frac{\pi}{4} + \int_{z_v}^{z_2} k_z dz \right] \quad , \tag{45b}$$

when $k_z^2(z_v) = 0$ for some $z_v > 0$ and $< z_2$. These expressions give the ray-acoustical modification to source strength consequent to taking the coherent sum of the contributions of the two rays that leave the source at angles $\pm\theta$. In Eq. 45a, the upgoing ray reflects from the surface; in Eq. 45b, the upgoing ray is refracted without reaching the surface.

These ray-acoustical results are to be compared with the familiar *Lloyd's mirror formula*, which is the precise solution in an environment having constant sound speed:

$$D_0^{LM} = 4 \, \sin^2[k_z(z_2)z_2] \quad . \tag{46}$$

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Horizontal Dipole

Consider now the radiation from a horizontal dipole source — that is, a fluctuating force F, oriented horizontally.

It is immediately evident from a comparison of the wave equations, Eqs. 3 and 4, that the field radiated from the dipole is related to the unit monopole field G by

$$p = \frac{F \cos\alpha}{4\pi} \frac{\partial G}{\partial r} \quad , \tag{47}$$

where α is the azimuth angle in the horizontal plane between the force vector and the line from the source to the receiver. But, the dominant dependence of G on r, in the far field, is in the phase (Eqs. 16 and 19):

$$G \propto \exp ik_v r \quad .$$

Hence, we have

$$p = \left[\frac{ik_v F \cos\alpha}{4\pi} \right] G \quad . \tag{48}$$

Now, the analysis for the power in a ray bundle exactly parallels the earlier analysis (Eqs. 33 to 38) for a unit monopole source. The result for the horizontal dipole is readily seen to equal the result for the unit monopole multiplied by the squared magnitude of the factor in brackets in Eq. 48. The effective source strengths are similarly related. Thus, the *effective source strength* of the horizontal force can be written as

$$S_{eff} = [k(z_2)|F|/4\pi]^2 D_{HF} \quad ,$$

$$D_{HF} = \cos^2\theta(z_2) \cos^2\alpha \, D_0 \quad , \tag{49}$$

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D_0 being given by Eq. 44. The ray-acoustical approximation for D_0 , Eq. 45, and Lloyd's mirror formula, Eq. 46, are equally applicable here.

Vertical Dipole

Consider now a vertical dipole (force) source of strength F. The comparison of Eqs. 3 and 4 shows that the radiated pressure is related to that for a unit monopole by

$$p = [F/4\pi] \, \partial G/\partial z_2 \quad . \tag{50}$$

The analysis for power in a ray bundle parallels that for a unit monopole. The result is readily seen to differ only by the squared magnitude of the factor $F/4\pi$ and by the substitution of $(\partial g_u/\partial z_2)$ for g_u. Thus, the *effective source strength* of the vertical force can be written as

$$S_{eff} = [k(z_2)|F|/4\pi]^2 \, D_{VF} \quad ,$$

$$D_{VF} = \frac{k_z(z_2) \, |\partial g_u(z_2,z_1,k_v)/\partial z_2|^2}{k_z(z_1) \, k^2(z_2)} \quad . \tag{51}$$

The ray-acoustical approximation for D_{VF} is found by differentiating the WKB expression for g_u, Eq. 22. For the case of rays that strike the surface $[k_z^2(0)>0]$, the result is found to be

$$D_{VF}^{WKB} \approx 4 \sin^2\theta(z_2) \cos^2\left\{\int_0^{z_2} k_z dz\right\} \quad , \tag{52}$$

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where we have omitted a term involving $(\partial k_z / \partial z_2)$ that is only important when the cosine is nearly zero. In the case analogous to Lloyd's mirror, when k_z is independent of depth in the layer, the ray-acoustical result is

$$D_{VF}^{LM} = 4 \sin^2\theta(z_2) \cos^2[k_z(z_2)z_2] \quad . \tag{53}$$

6.   RANGE-AVERAGED MODIFIED RAY THEORY

Some of the errors of ordinary ray theory have little or no effect when the predictions are averaged over an interval of the range variable.

Consider the predictions of the sound field by both ordinary and modified ray theory when the radiation is restricted to a ray bundle dk_V centered on k_V. Both theories yield similar results, as sketched in Fig. 4. At a fixed receiver depth z_1, the ray bundle irradiates a small range interval on the way down, and another small interval at a slightly greater range on the way up. (These correspond to the terms G_n^+ and G_n^- in the multipath expansion for the monopole field.) Then, at a range interval equal to the period X, the pattern is repeated. (These correspond to the terms G_{n+1}^+ and G_{n+1}^-.)

Differences between ordinary and modified ray theories are exhibited in the proportions and spacing of the irradiated zones. The width of each zone is proportional to the range derivative, being given by

$$dr = |\partial r / \partial k_V| \, dk_V \quad .$$

The range derivative has different values in the two theories. Moreover, the squared pressure in each zone is inversely proportional to the range derivative (Eqs. 18 and 37), so that it also differs between the two theories, but inversely to the difference in zone width. The result suggests that we should consider the range-averaged prediction.
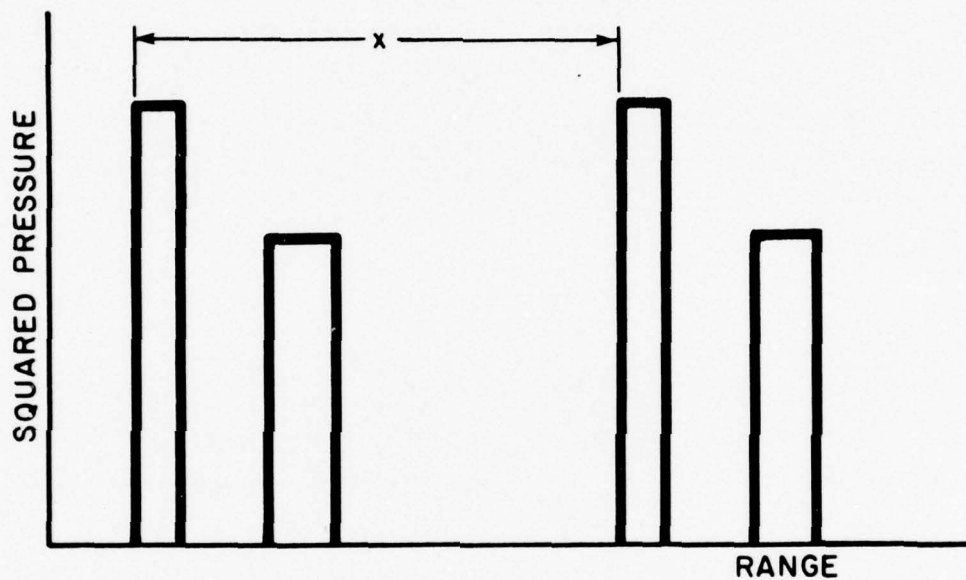
FIG. 4.   THE PATTERN OF SQUARED PRESSURE AS A FUNCTION OF RANGE, WHEN ONLY
A SMALL RAY BUNDLE IS EXCITED. THE PATTERNS PREDICTED BY ORDINARY
AND MODIFIED RAY THEORIES ARE SIMILAR BUT DIFFER IN DETAIL (SEE TEXT).

Following procedures earlier found to be convenient in ordinary ray theory [13], we shall average the contribution of each ray bundle over a range interval equal to the range period X, taking all ranges to be equally probable. Then, we sum these averaged contributions over all possible ray bundles. For simplicity of exposition, we neglect attenuation and set $|V_u V_\ell| = 1$.

First, we need an expression for the squared pressure observed in the zone of range that is irradiated by a ray bundle. By the same algebra that was used earlier for monopole radiation (Eqs. 33 through 35), we calculate for a general source the squared pressure observed at depth z :

$$p^2 = \rho c(z_1) I = \rho \omega I / k(z_1)$$

$$= \frac{\rho \omega}{k(z_1)} \frac{\partial P / \partial k_v}{\partial A / \partial k_v} = \frac{\rho \omega}{k_z(z_1)} \frac{\partial P / \partial k_v}{2\pi r |\partial r / \partial k_v|}\bigg|_{z_1} \quad ,$$

where I is intensity, P is power, and dA is the transverse area of a ray bundle dk_v. Now, in Sec. 5, the effective source strength, S_{eff}, of any source was defined as the source strength that yielded the correct value for power in a ray bundle when used in the ordinary ray-acoustical formula (Eq. 41):

$$\partial P / \partial k_v = 2\pi k_v S_{eff} / [\rho \omega k_z(z_2)] \quad ,$$

where z_2 is the source depth. The combination of these results is:

$$p^2(z_1) = \frac{k_v S_{eff}}{r k_z(z_1) k_z(z_2) |\partial r / \partial k_v|_{z_1}} \quad . \tag{54}$$

To form the average contribution of any ray bundle dk_V, we multiply p^2 by the ratio of the irradiated zone width to the range period $X(k_V)$, a factor

$$\frac{|\partial r/\partial k_V|\ dk_V}{X} \quad .$$

The result is seen to be the same for each of the two zones (corresponding to multipath terms G_n^+ and G_n^-) that occur in each range period. (The slight variation in r is ignorable at long ranges.) Adding these two contributions and summing over all ray bundles, one finds the average squared pressure

$$<p^2(z_1)> = \int \frac{2\ k_V\ S_{eff}}{r\ X\ k_Z(z_1)k_Z(z_2)}\ dk_V \quad , \tag{55}$$

where the integral extends over all k_V such that the modified ray reaches the depth z_1 (i.e., $k_Z^2(z_1)>0$).

It is often convenient to introduce as a new integration variable, in place of k_V, the ray-acoustical inclination angle at the source, θ_2, defined by Snell's law:

$$k_V = k(z_2)\ \cos\theta_2 \quad , \quad k_Z(z_2) = k(z_2)\ \sin\theta_2 \quad . \tag{56}$$

Then Eq. 55 can be written in the form

$$<p^2(z_1)> = \int_{\theta_2>0} \frac{2\ S_{eff}}{r\ X\ \tan\theta_1}\ d\theta_2 \quad , \tag{57}$$

where θ_1 is the inclination angle at the receiver's depth z_1 as computed by Snell's law. This integral is restricted to

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positive θ_2, since the contributions of what would in ordinary ray acoustics be rays at $\pm\theta_2$ are both included in the effective source strength. The contribution of k_v corresponding to imaginary values of θ_2 can, as a practical matter, be excluded without significant error because their contribution is much weaker than that of the real angles.

The result is identical in form with the corresponding result of ordinary ray theory [Ref. 13, Eq. 4]. However, the value of the integrand differs from the value computed from ordinary ray theory for two reasons. First, the range period X is not the same in modified as in ordinary ray theory. Based on the illustrative results of Murphy and Davis [11] and other data presented in this report, this error appears typically to be small and negligible. The value of X given by ordinary ray acoustics is sufficiently accurate for a range-averaged modified ray theory.

The second difference lies in the value of the effective source strength S_{eff}. As shown by the analytical results presented elsewhere in this report, ordinary ray theory can lead to serious errors in evaluating it.

By using the effective source strength, found by a wave analysis in the surface layers only, methods of ordinary ray theory can be used to calculate an accurate value for the range-averaged transmission, although the point-to-point transmission will be inaccurate. Of course, there is no need to perform the point-to-point calculations when the averaging can be done analytically, with calculations based on Eq. 57.

7.   ISOGRADIENT SURFACE LAYERS

In this section, analytical expressions for the wave-theoretical modifications to ray theory are developed for the case of a source in an isogradient surface layer. It is assumed throughout that the layer is deep enough that ordinary ray (WKB) analysis suffices to trace the spreading of sound energy below the layer.

Practitioners of wave theory will have guessed that the problem we shall solve is that for which the Airy functions are precise solutions to the one-dimensional wave equation. We postulate an environment in which the gradient with depth of k^2 is a positive constant:

$$dk^2(z)/dz = B^3 \quad . \tag{58}$$

Since $k = \omega/c$, this prescription implies that the sound-speed gradient is negative and given by

$$g^* = -dc/dz = \pi f B^3/k^3 \quad , \tag{59}$$

where $f = \omega/2\pi$ is the frequency. Since k varies with depth, the sound-speed gradient is not strictly constant in the layer. However, its variations are negligible in practical cases.

We require solutions to the one-dimensional, separated wave equation (Eq. 8):

$$d^2g(z)/dz^2 + k_z^2(z) \, g(z) = 0 \quad , \quad z \geq 0 \quad ,$$

$$g(0) = 0 \quad ,$$

$$k_z^2 = k^2(z) - k_v^2 \quad , \tag{60}$$

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where the radial wavenumber k_v is constant. The transformation
of coordinates

$$y = B(z-z_v) \quad ,$$

$$k_z^2 = B^3(z-z_v) = B^2 y \quad , \tag{61}$$

brings the wave equation into Airy's form:

$$d^2 g(y)/dy^2 + y \, g(y) = 0 \quad , \; y \geq y_0 \quad ,$$

$$g(y_0) = 0 \quad , \tag{62}$$

where y_0 is the value of y corresponding to the ocean surface,
$z = 0$. Note in Eq. 61 that z_v is the depth corresponding to the
ray vertex, or turning point, where k_z vanishes. Its value may
be positive or negative, corresponding in the latter case to the
depth at which the ray would have vertexed if the layer had ex-
tended above the actual sea surface.

The solution to this boundary valve problem is well known:

$$g(y) = C \, M(y) \, \sin[\psi_0 - \psi(y)]$$

$$\psi_0 = \psi(y_0) \quad , \tag{63}$$

where C is an arbitrary constant and M and ψ are real functions
defined in terms of Airy functions:

$$M(y)e^{i\psi(y)} = Ai(-y) + i \, Bi(-y) \quad . \tag{64}$$

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Here, Ai and Bi are the standard Airy functions, for which various analytical properties and tables of values are found in standard texts [Abramowitz and Stegun, Ref. 14]. Equation 63 is the linear combination of Airy functions that matches the boundary condition at $y = y_0$.

Normalization

In the preceding development of modified ray theory, the solution to the one-dimensional wave equation was normalized in a very particular way. Specifically, the desired solution is the response to an upwardly traveling wave that has unit amplitude and zero phase at a reference depth z_1. (See Eqs. 7 and 9.) The reference depth is chosen to be sufficiently deep that the asymptotically precise, ordinary WKB expression accurately describes the local field, but it is otherwise arbitrary. That is, the constant C in Eq. 63 is to be chosen so that, for z and z_1 both deep in the asymptotic region,

$$g = g_u(z,z_1) = \frac{2\, k_z^{\frac{1}{2}}(z_1)}{k_z^{\frac{1}{2}}(z)} \times$$

$$\sin\left[\frac{1}{2}\phi_u + \int_{z_1}^{z} k_z dz\right] e^{i(\frac{1}{2}\phi_u - \frac{\pi}{2})} \quad . \tag{65}$$

(Compare Eq. 21.) Here, ϕ_u has the significance that

$$V_u(z_1) = -e^{i\phi_u} \tag{66}$$

is the plane-wave reflection coefficient of the ocean, $0 < z < z_1$, as observed at the depth z_1.

The asymptotic forms of M and ψ for large positive y are
[Ref. 14; Eqs. 10.4.78, 10.4.79]:

$$M \sim \pi^{-\frac{1}{2}} y^{-\frac{1}{4}}$$

$$\psi \sim \frac{1}{4}\pi - (2/3)y^{3/2}$$

$$d\psi/dy \sim -y^{\frac{1}{2}} \quad , \qquad\qquad (67)$$

which are accurate for $y^3 \gg 5/32$. These forms can be translated
by means of Eq. 61 into:

$$M \sim (B/\pi k_z)^{\frac{1}{2}}$$

$$\psi(y) \sim \psi_1 - \int_{z_1}^{z} k_z dz$$

$$\psi_1 = \psi(y_1) \quad , \qquad\qquad (68)$$

where y_1 is the value of y corresponding to $z = z_1$. Now, a direct
comparison of the desired form, Eq. 65, with the general Airy
function solution, Eq. 63, yields the required value of the con-
stant:

$$C = 2\pi^{\frac{1}{2}} B^{-\frac{1}{2}} k_z^{\frac{1}{2}} (z_1) e^{i(\frac{1}{2}\phi_u - \frac{1}{2}\pi)}$$

$$\phi_u = \arg[-V_u(z_1)] = 2(\psi_0 - \psi_1) \quad . \qquad\qquad (69)$$

Error in Range Period

In this subsection, we evaluate the error in range period X that arises from the use of ordinary ray acoustics.

The range period is related to the phases of the plane-wave reflection coefficients for the upper and lower halves of the ocean (Eq. 32). In this study, we are concerned with the error arising from the upper half. The error in the range period is due to the difference between the wave-theoretical phase and the asymptotic expression for that phase, since the asymptotic expression is that used in ordinary WKB analysis. The phase of the reflection coefficient in the present isogradient problem is given in Eq. 69. Only the term $\psi_0 = \psi(y_0)$ differs from its asymptotic form, since the observation depth z_1 is by definition so large that ψ_1 is given by its asymptotic value. Thus, we are led, by the combination of Eqs. 32 and 69, to this expression for the error in the range period (i.e., the ray-acoustical value less the wave-acoustical value):

$$\Delta X = -2 \; \partial[\psi^{as}(y_0) - \psi(y_0)]/\partial k_V \quad , \tag{70}$$

where ψ^{as} denotes the asymptotic form of ψ, and y_0 is implicitly a function of the ray parameter k_V.

Two asymptotic forms must be distinguished, depending on the sign of y_0:

$$\psi^{as}(y) = \begin{cases} \pi/4 - 2y^{3/2}/3 & \text{for } y > 0 \\ \pi/2 & \text{for } y < 0 \end{cases} \quad . \tag{71}$$

The first has been given in Eq. 67. The second follows from the known asymptotic behavior [14]:

$$Ai(x)/Bi(x) \sim 0 \quad \text{as } x \sim \infty .$$

The derivative in Eq. 70 can usefully be transformed by successive application of Eqs. 60, 61, and 59:

$$\frac{\partial}{\partial k_v} = 2 k_v \frac{\partial}{\partial (k_v^2)} = -2 k_v \frac{\partial}{\partial (k_v^2)} = -\frac{2 k_v}{B^2} \frac{d}{dy} \quad .$$

Then, the error in range period takes the form

$$\Delta X = \frac{4 k_v}{B^2} \left\{ \psi'(y_0) + \begin{bmatrix} y_0^{\frac{1}{2}} & , & y_0 > 0 \\ 0 & , & y_0 < 0 \end{bmatrix} \right\} , \tag{72}$$

where $\psi' = d\psi/dy$.

The environmental variables and ray parameters are rather intricately interwoven in the variables and parameters that appear in Eq. 72. However, it is possible to show that the error in wavelengths is a function of only two variables: (i) the ratio of frequency to sound-speed gradient, and (ii) a ray-theory angle at the surface defined by

$$k_v = k(0) \cos\theta_0 \quad . \tag{73}$$

This angle is imaginary for rays that vertex below the surface although it is still a useful variable, being then related to the depth of the turning point expressed in units of the wavelength. Graphs of typical values are shown in Figs. 5 and 6.
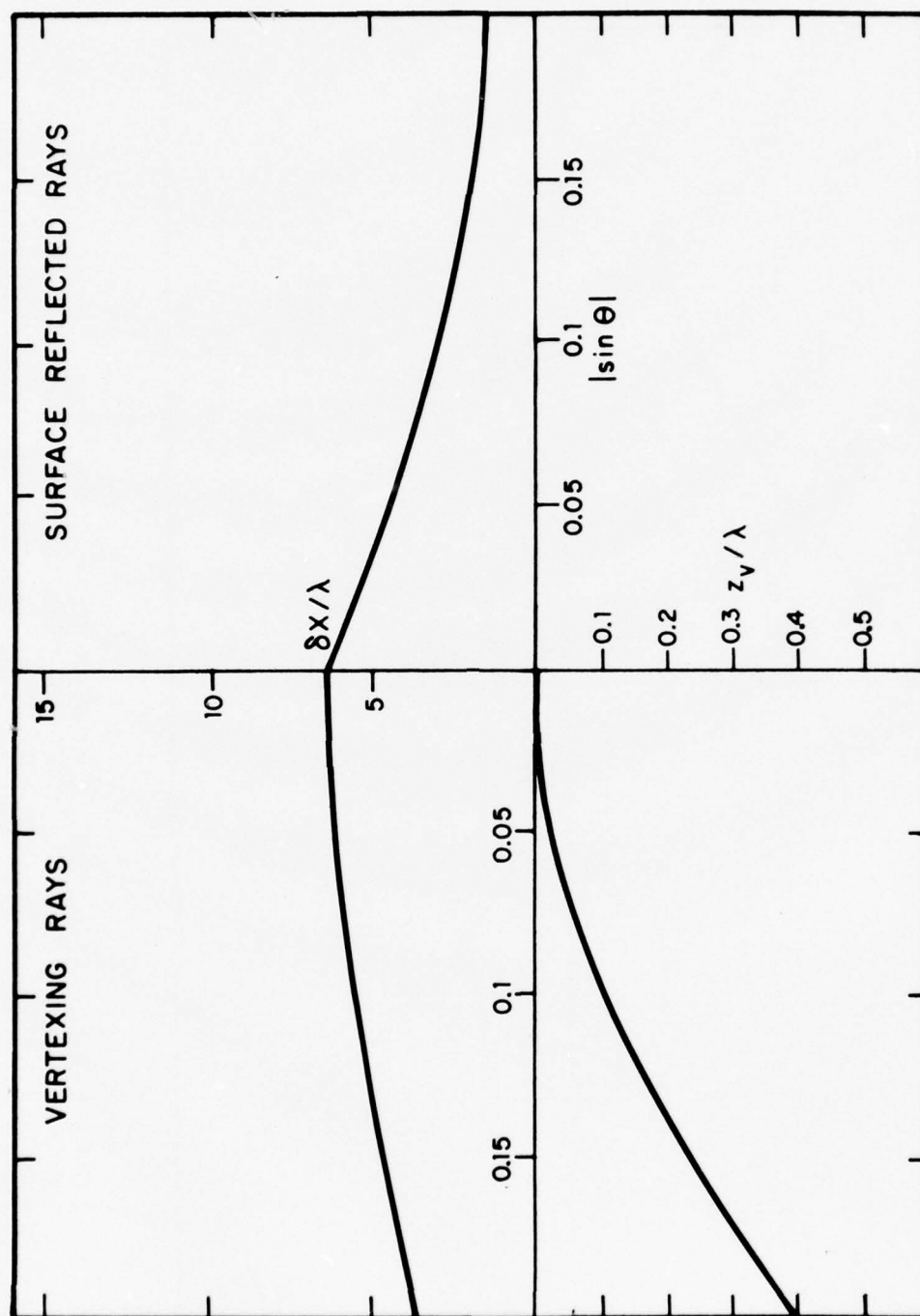
FIG. 5. ERROR δX IN THE RAY-THEORETICAL RANGE PERIOD AS A FUNCTION OF $|\sin\theta|$ AT THE SURFACE, FOR $(f/q^*) = 20$. THE LEFT HALF OF THE GRAPH APPLIES TO ORDINARY RAYS THAT VERTEX AT A DEPTH z_x BELOW THE SURFACE, FOR WHICH θ IS IMAGINARY.

FIG. 6. ERROR δX IN THE RAY-THEORETICAL RANGE PERIOD AS A FUNCTION OF $|\sin\theta|$ AT THE SURFACE, FOR $(f/q^*) = 50$. THE LEFT HALF OF THE GRAPH APPLIES TO ORDINARY RAYS THAT VERTEX AT A DEPTH z_X BELOW THE SURFACE, FOR WHICH θ IS IMAGINARY.
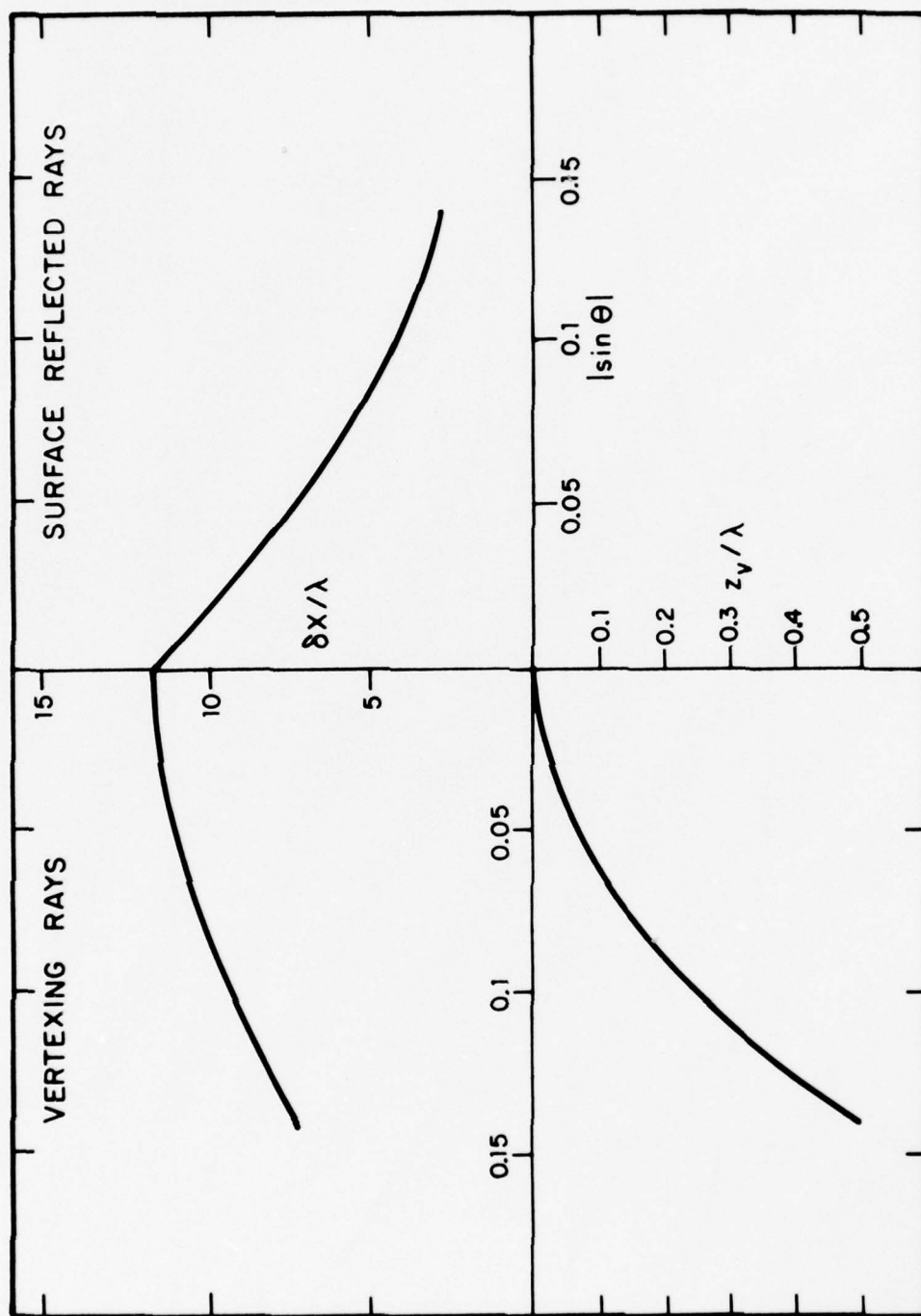
It will be noted that the largest error in the range period occurs for the ray whose turning point is at the surface. This maximum error is given by

$$\Delta X_{max} = \frac{1.261}{f^{1/3} \, g^{*2/3}} \text{ km} \quad , \tag{74}$$

where f is in Hz, g^* in sec^{-1}, and the value 1462 m/s has been used for sound speed. This relation is plotted in Fig. 7.

Equation 74 and Fig. 7 are both very deceptive. Two cautions must be stressed. First, the analysis assumes that the constant-gradient layer is infinitely deep. It is applicable to a layer of finite depth only if the ray in question (horizontal at the surface) steepens within the layer to an extent that asymptotic analysis is valid at greater depths. Taking values of y > 1 as defining the asymptotic, ordinary WKB region, one can calculate the layer depth required for validity of the results. The limiting contours in the case of layer depths of 50 and 100 m have been plotted in Fig. 7.

The second caution concerns the change in range-period error with changes in the gradient, frequency being held constant. Figure 7 shows the error *increasing* as the gradient *decreases*! The result seems to contradict our intuition, which is based on the expectation of small errors in ray theory when the gradient is small. The paradox is resolvable. First, the limit as gradient approaches zero is not pertinent to physical problems, because of the limited depth of actual isogradient layers. Second, the rate of change of error with gradient depends on the steepness of the ray, as a comparison of Figs. 5 and 6 will indicate. Only for a
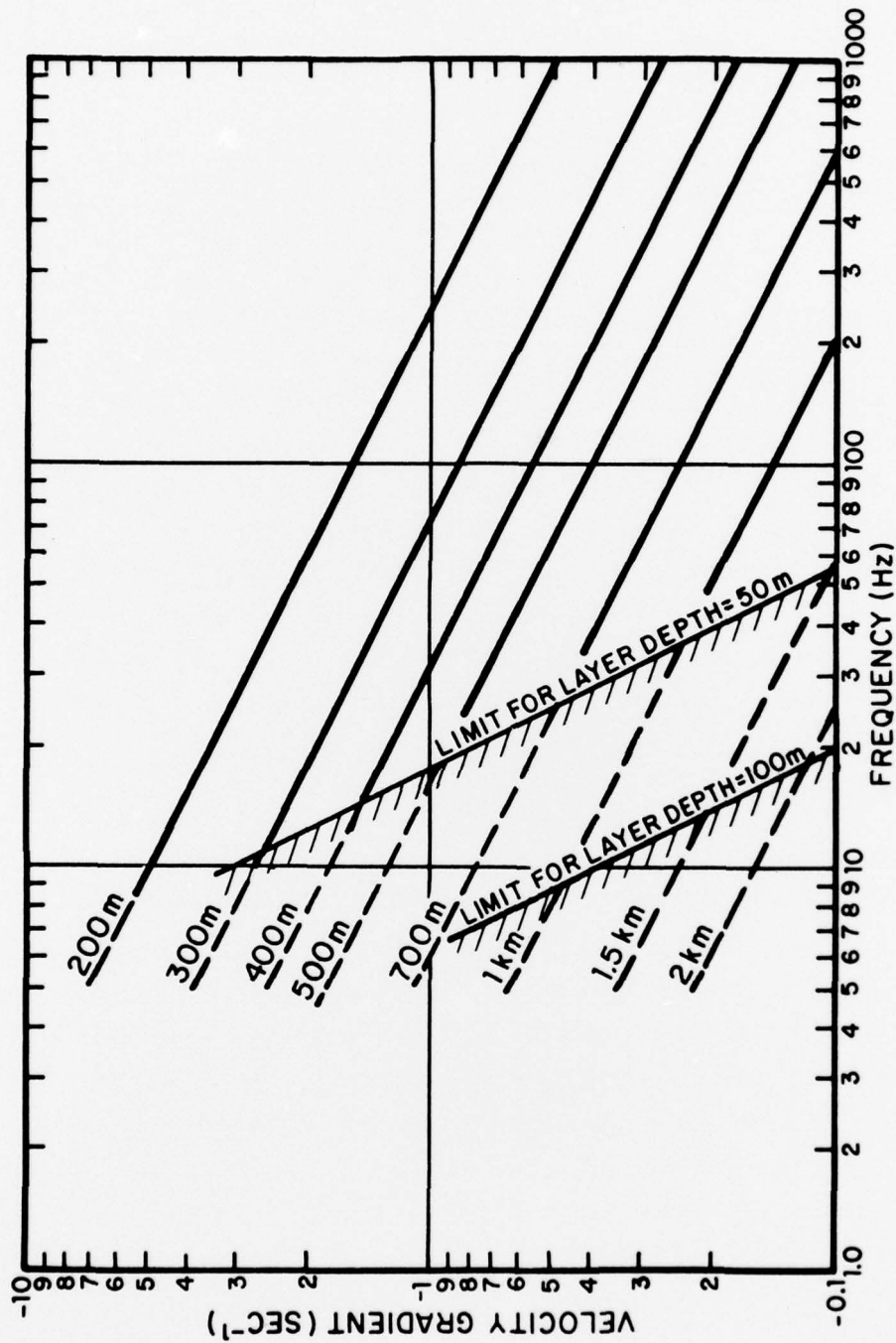
FIG. 7.   CONTOURS OF MAXIMUM ERROR IN THE RAY-THEORETICAL RANGE PERIOD.   THE MAXIMUM OCCURS FOR THE RAY HORIZONTAL AT THE SURFACE.   THE RESULTS, VALID FOR VERY DEEP LAYERS, ARE VALID ONLY TO THE RIGHT OF THE LIMITS SHOWN FOR FINITE LAYER DEPTHS.

narrow beam of rays, centered on the ray that grazes the surface, does the error grow with decreasing gradient. The width of that beam becomes smaller as gradient decreases or as frequency increases. Eventually, the beam width becomes insignificant. However, Figs. 5 and 6 demonstrate that a significant beam has this anomalous behavior at low frequencies.

Effective Source Strength of Monopole

The general formula for effective source strength of a monopole (volume source) is given in Eq. 44. For a unit monopole ($S_0 = 1$), the effective source strength D_0 involves the solution to the separated wave equation, g_u, evaluated at source depth z_2. The suitably normalized function g_u for the isogradient environment is given by Eq. 63 with the constant C given in Eq. 69. The combination of these results can be written as

$$D_0 = 4\pi[y^{\frac{1}{2}}M^2 \sin^2(\psi_0-\psi)]_{y=y_2} \quad , \tag{75}$$

where y_2 is the value of y at $z = z_2$. (The explicit appearance of the environmental parameter B has been eliminated from Eq. 75 by use of Eq. 61.)

The variable y is dependent on frequency, sound-speed gradient, and a variable identifying the ray. It is convenient to use as ray variable the inclination angle θ_2 at source depth, as given by Snell's law (Eq. 56). Then, using Eqs. 59 and 61, one can show that

$$y_2 = k_z^2(z_2)/B^2 = (\pi f/g^*)^{2/3} \sin^2\theta_2 \quad , \tag{76}$$

where g^* is the negative of sound-speed gradient and f is frequency.

This wave-theoretical value of D_0 is to be compared with the ordinary WKB, ray-theoretical value compared from Eq. 45, and with Lloyd's mirror formula, Eq. 46. Figure 8 shows this comparison for typical values of the parameters. It is interesting to see that a precise application of ordinary ray theory is less accurate than Lloyd's mirror formula. The wave-theoretical result (labeled "modified ray theory") indicates that the effective source strength is significantly larger than the approximations for it, when the angle θ_2 is small.

The general functional dependence of D_0 on θ_2, involving the functions $M(y)$ and $\psi(y)$, is very intricate. However, a simple approximation can be derived for the cases of principal interest. First, we are concerned with sources close to the ocean surface, in the "decoupling layer", where the sine function in Eq. 75 is small and its argument is small relative to $\pi/2$. Indeed, one can show that the asymptotic, ordinary WKB solution is good when this is not the case. Then, the sine is approximately equal to its argument, which can be rewritten using the law of the mean:

$$\sin(\psi_0 - \psi) \approx \psi_0 - \psi = \psi'(\overline{y}) \ (y_0 - y)$$

where ψ' is the derivative of ψ and $\overline{y}$ is a mean value of the argument, $y_0 \leq \overline{y} \leq y$. But the derivative of the phase, ψ', is related to the amplitude M by an identity for Airy functions [Ref. 14, Eq. 10.4.71]:

$$\psi'(y) = -1/\pi M^2(y) \quad .$$

We introduce these approximations into Eq. 75. Further, $(y_0 - y_2)$ is related to z_2 by Eq. 61, and $y_2^{\frac{1}{2}}$ to $\sin\theta_2$ by Eq. 76. Finally,
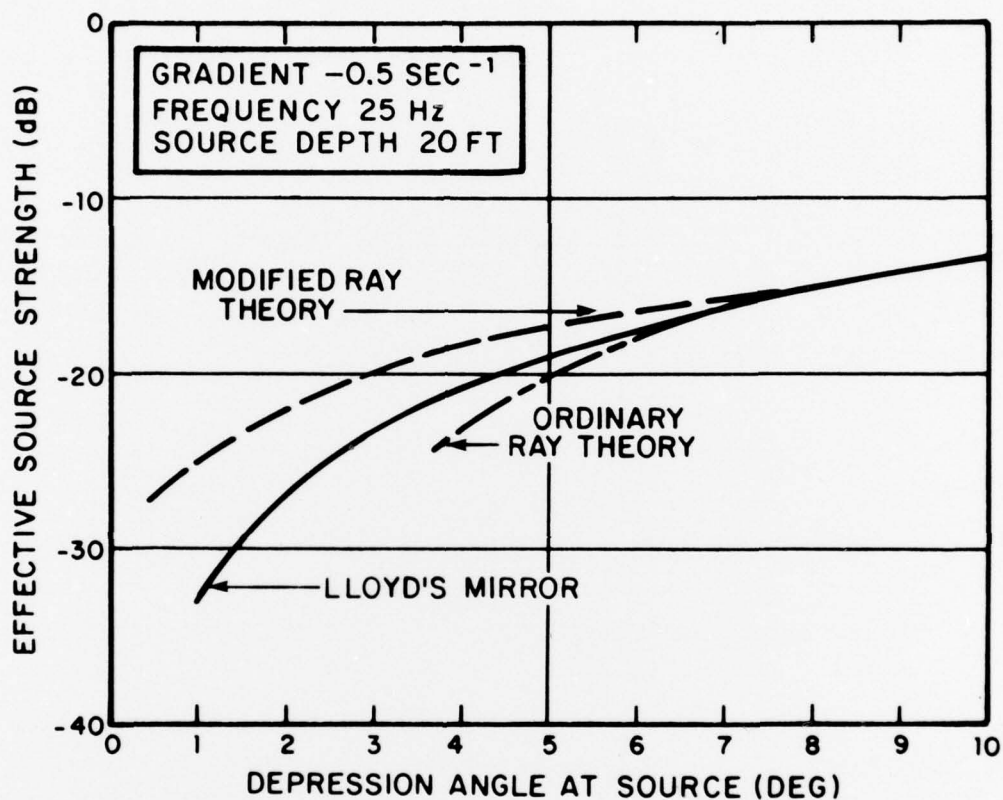
FIG. 8.   EFFECTIVE SOURCE STRENGTH D_o OF A UNIT MONOPOLE SOURCE, CALCULATED FROM EQ. 75.

we introduce the approximation

$$M^2(y_2)/M^4(\overline{y}) \approx 1/M^2(0) = 1.983 \quad ,$$

which is accurate when both y_2 and y_0 are small — that is, in the region where the wave-theoretical correction is important. The resulting approximation for effective source strength is

$$D_0 \approx 68 \left(\frac{g^*}{f}\right)^{1/3} \left(\frac{z_2}{\lambda}\right)^2 \sin\theta_2 \quad . \tag{77}$$

The range of validity of the approximation is derived below (Eq. 79).

This approximation is to be contrasted with a similar approximation to Lloyd's mirror formula (Eq. 46):

$$D_0^{LM} = 4 \sin^2(k_z z_2) \approx 4(k_z z_2)^2$$

or, writing $k_z = k \sin\theta_2$,

$$D_0^{LM} \approx 158(z_2/\lambda)^2 \sin^2\theta_2 \quad . \tag{78}$$

The result is valid when $D_0^{LM} \ll 4$, which is not a serious limitation in the cases of present interest.

The striking difference between Eqs. 77 and 78 lies in their dependence upon θ_2. This difference causes the error in the Lloyd's mirror formula to grow with decreasing values of inclination angle θ_2. Figure 9 shows the results of precise calculations of the error as a function of θ_2 and the ratio of frequency to gradient. Also plotted on the figure is the estimate of the error formed by the ratio of Eq. 78 and Eq. 77. Within the precision of these approximations, the error is independent of source depth.
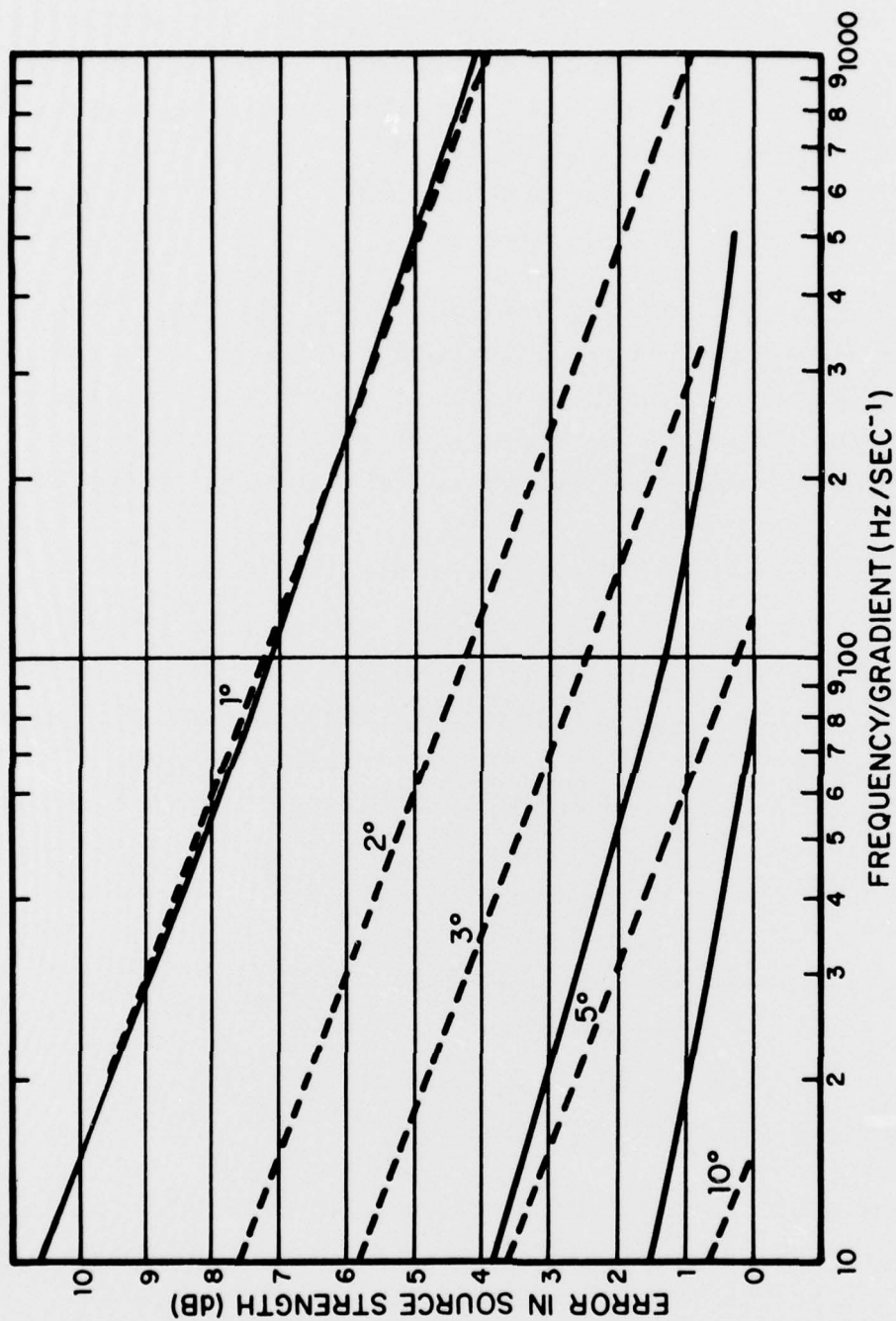
FIG. 9.   ERROR IN THE EFFECTIVE SOURCE STRENGTH OF A UNIT MONOPOLE SOURCE THAT RESULTS FROM LLOYD'S MIRROR FORMULATION. THE DASHED LINES ARE APPROXIMATIONS CALCULATED FROM EQS. 77 AND 78.

Finally, the limit of validity of the approximate wave-theoretical result, Eq. 76, can be established from the observation that the approximation is good when it exceeds the Lloyd's mirror formula. Hence, Eq. 77 is valid if

$$g^*/f > 12.5 \sin^3\theta_2 \ , \tag{79}$$

which is equivalent to $y_2 < 0.40$ (see Eq. 77). When this inequality fails, the Lloyd's mirror formula is accurate, as is shown by the convergence of the precise values in Fig. 8. When this inequality holds, both the Lloyd's mirror formula and the precise result of ordinary ray theory are inaccurate.

Effective Source Strength of Vertical Dipole

The general formula for the effective source strength of a vertical dipole (force source) is given in Eq. 51. For a unit force F, the effective source strength D_{VF} involves the spatial derivative of the solution to the separated wave equation, evaluated at source depth. With the introduction of the solution for the isogradient environment (Eqs. 63 and 69) into Eq. 51, the effective source strength per unit force can be written as

$$D_{VF} = 4\pi[k_z(z_2)/k(z_2)]^2 \ [y^{-\frac{1}{2}}N^2\sin^2(\psi_0-\eta)]_{y=y_2} , \tag{80}$$

where

$$N(y)e^{in(y)} = Ai'(-y) + i\ Bi'(-y) \ . \tag{81}$$

This wave-theoretical value of D_{VF} is compared with the ordinary WKB, ray-theoretical value (Eq. 52) and the Lloyd's mirror formula (Eq. 53) in Fig. 10. As was the case for the
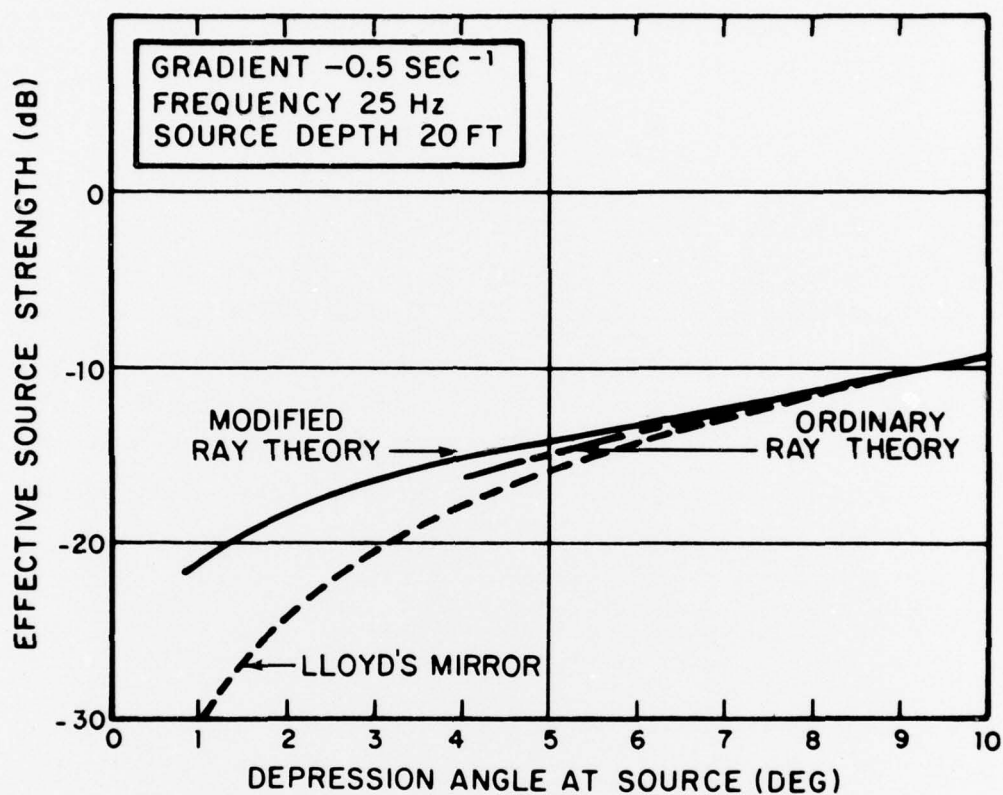
FIG. 10. EFFECTIVE SOURCE STRENGTH D_{VF} OR A VERTICAL DIPOLE, CALCULATED FROM EQ. 80.

monopole source, the wave-theoretical value of source strength
(labelled "modified ray theory") is significantly larger than the
approximations when the ray inclination angle is small.

An approximation to Eq. 80 displays the explicit dependence
on physical parameters. One starts with the expansion

$$\psi_0 - \eta = (\psi_0 - \psi) - (\eta - \psi) \quad .$$

Numerical experience indicates that the first term on the right
is small and negligible with respect to the second term in the
range of parameters of interest. The second term ranges from
$\pi/3$ to $\pi/2$ for all $y \geq 0$ ($\theta_2 \geq 0$). But the first term, which
appears in Eq. 75 for the monopole, is small relative to $\pi/2$ in
the surface decoupling layer. The sine of the second term is
now transformed by an identity for Airy functions [Ref. 14, Eq.
10.4.74],

$$\sin(\eta - \psi) \equiv 1/\pi \ MN \quad ,$$

with the result

$$D_{VF} \approx 4[k_z(z_2)/k(z_2)]^2 \ [\pi \ y_2^{\frac{1}{2}} M^2(y_2)]^{-1}$$

$$= 4 \sin^2\theta_2 [\pi y_2^{\frac{1}{2}} M^2(y_2)]^{-1} \quad . \tag{82}$$

It is interesting to note the asymptotic behavior of Eq. 82.
From the asymptotic expression for M (Eq. 67), which is a good
approximation for $y_2 \geq 1$, one finds

$$D_{VF} \approx 4 \sin^2\theta_2 \quad ,$$

which is identical with an approximation of the Lloyd's mirror formula (Eq. 53):

$$D_{VF}^{LM} = 4 \sin^2\theta_2 \cos^2(k_z z_2) \approx 4 \sin^2\theta_2 \ . \tag{83}$$

(The approximation, good for $k_z z$ small, is not an important limitation in the practical problems of interest.) Hence, we have demonstrated analytically the confluence of D_{VF} and D_{VF}^{LM} for $y_2 > 1$, which is observed in the numerical data of Fig. 10.

The asymptotic form for M^2 becomes significantly poor (errors of 1 dB or more) for values of $y_2 < 0.5$. However, the value of M^2 does not vary widely in the interval $0.5 \geq y_2 \geq 0$, so that the geometric mean value

$$\overline{M}^2 = [M^2(0) \ M^2(0.5)]^{\frac{1}{2}} = 0.432$$

can be used without introducing errors greater than 0.7 dB. Using this approximation in Eq. 82 and introducing the physical parameters, we find the approximation

$$D_{VF} \approx 2.01 \ (g*/f)^{1/3} \sin\theta_2 \quad . \tag{84}$$

This approximation is good for real θ_2 so long as it exceeds the asymptotic expression previously shown to be the same as Eq. 83. Hence, Eq. 84 is applicable if

$$(g*/f) > 7.9 \sin^3\theta_2 \ , \tag{85}$$

which is equivalent to $y_2 < 0.54$ (see Eq. 77).

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As in the monopole case, we find from wave theory that the effective source strength varies as sinθ when θ approaches zero.

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    Propagation Effects for Sources or Receivers Near the Ocean
    Surface," NUC Report TP 488, September 1975.

2. E.L. Murphy and J.A. Davis, "Modified Ray Theory for Bounded
    Media," *J. Acoust. Soc. Am.* **56**, 1747-1760 (1974).

3. It is relatively easy to include variations of density. As
    shown by Bergmann (Ref. 4, p. 171), the transformation
    $p = \rho^{\frac{1}{2}}\psi$ leads to an ordinary wave equation for ψ which has a
    frequency dependent index of refraction. Moreover, the source
    term for a force source can be shown to include both a mono-
    pole and a dipole component. The consequent effects are
    universally ignored in the literature, and they would not
    affect the essential results to be developed here.

4. L.M. Brekhovskikh, *Waves in Layered Media*, Academic Press,
    New York, 1960.

5. Ref. 4, §36.

6. Ref. 4, §38.

7. The mathematical justification for using this formula must
    be based on the asymptotic behavior of the integral as some
    parameter increases. The customary parameter, frequency,
    is more than a little awkward, first because it leads
    asymptotically to ordinary ray acoustics, and secondly because
    increasing frequency introduces rapid fluctuations in the
    magnitude of g_u due to interference of direct and surface-
    reflected waves. Range is a more attractive large parameter.
    The procedure then involves considering a sequence of in-
    tegrals, with increasing n, evaluated at a sequence of
    ranges such that the stationary phase point is constant.
    Since we are interested in ducted (low loss) energy, we may
    reasonably set $|V_u V_\ell| = 1$; the stationary phase expression
    is asymptotically precise in this sequence. More generally,
    for losses not negligible, we should have to use the nearly
    equivalent method of steepest descent.

8. The generalization has its cost. The "effective strength"
    of the source, defined later, has different values for dif-
    ferent receiver depths.

9. For example, Ref. 4, §16.

10. I. Tolstoy and C.S. Clay, *Ocean Acoustics*, McGraw-Hill Book Co., Inc., New York, New York, 1966; Sec. 2.5.

11. Reference 2, Fig. 12.

12. The situation would be considerably more complicated if the environment included several WKB regions interspersed in depth with several non-WKB regions.

13. P.W. Smith, Jr., "Averaged Sound Transmission in Range-Dependent Channels," *J. Acoust. Soc. Am.* **55**, 1197-1204 (1974). The considerations in this reference of range-dependent environments should also carry over into the present analysis.

14. M. Abramowitz and I.A. Stegun (Eds.), *Handbook of Mathematical Functions*, U.S. Government Printing Office, Washington, D.C., 1964; Chap. 10.